Profitability of Frequency Reward Programs

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Abstract

We study the profitability of frequency reward programs under very general conditions. We begin by comparing the profitability of a monopolist offering a buy-X-get-one-free program to that of spot prices. The reward program can never outperform spot prices, even though they converge in the limit as $X = +\infty$. This result is robust to a buy-X-get-Y-free program, flexible cost structures, credit-specific pricing, and most importantly, fully-flexible specifications of customer heterogeneity. We then consider a duopoly in which both firms choose simultaneously between offering a reward program and spot prices. The only Nash equilibrium is for both firms to offer spot prices. Finally, we show that a reward program can outperform spot prices if credits have finite expiration. We provide intuition and managerial implications of these results and reconcile previous findings of profit-enhancing reward programs.

Keywords: customer reward programs, pricing, dynamic competition, switching costs, dynamic programming

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1 Introduction

Frequency reward programs are ubiquitous in a broad array of industries, including airlines, hotels, groceries, retailers, and coffee shops. The defining feature of a frequency reward program is that it rewards past purchases by basing prices on cumulative historical purchases; a typical structure is a buy-X-get-one-free program. Although these programs have been widely studied in marketing and economics, results on whether they are profitable marketing tools rely on specific contexts such as behavioral preferences, particular competitive landscapes, and finite credit expiration (Kim et al., 2001; Lu and Moorthy, 2007; Rossi, 2018; Sun and Zhang, 2019). Most other studies examine the design properties of reward programs conditional on their adoption (see Bombaij and Dekimpe (2020) for a survey). What remains under-studied, as far as we know, is the profitability of a frequency reward program in a simple but general setting with forward-looking customers who are infinitely lived. An infinite horizon fully accommodates the inherently dynamic decisions of customers facing a reward program. This would serve as a benchmark by which to understand the profitability of frequency reward programs under general conditions and, as a point of comparison, help to isolate why frequency reward programs are profitable in specific contexts.

In this paper, we analytically examine firms' incentives to introduce a reward program in an infinite-horizon setting with very general customer heterogeneity and endogenous purchase frequency. We consider both a monopoly and a duopoly. Specifically, we ask whether firms can realize higher profits by launching a reward program rather than simply charging standard, period-by-period spot prices? We begin by examining a monopolist with zero marginal cost offering a buy-one-get-one-free (B1G1) program. We consider the steady state of an infinite-horizon model with heterogeneous and forward-looking customers, which allows for rich patterns of heterogeneity in customers' valuations, time preferences, and purchase frequencies. Although customers' time horizons are infinite, the model is equivalent to customers having finite, but uncertain, shopping lifespans. The key finding is that the reward program cannot outperform period-by-period pricing. The under-performance of a reward program relative to spot prices extends to a fully-flexible specification of customer heterogeneity (a mixture of distributions), which rules out price discrimination between frequent and infrequent customers as a motivation for reward programs. It also holds under a very general specification of rules for earning and redeeming rewards (a buy-X-get-Y-free (BXGY) program), even though a reward program converges to spot prices when $X \to +\infty$ in a BXG1 program.

The reward program underperforms due to a "down-payment" effect. In a reward program, customers pay for future consumption in advance. This future consumption faces a time discount, lowering demand and firm profits. The intuition is most easily seen by considering a B1G1 program in which customers' valuations are certain. Suppose customers pay a price p to purchase and earn a credit, and 0 when they redeem a credit. In this scenario, they have to buy two units immediately but consume only one today. The willingness to pay for the unit consumed in the future is discounted. If the firm instead switches to a spot price of $0.5p$, customers consume as they pay with no time discounting. This increases their demand and, as a result, firm profits.[1] This intuition holds even when there is uncertainty in customers' future valuations defined under very weak distributional assumptions, including those commonly used in empirical work.

This intuition, and therefore the results, extend to frequency reward programs with other commonly observed features, including a non-zero marginal cost and credit-specific pricing in which the reward units are discounted rather than free. Any of the model features (credit-specific pricing, BXGY program, non-zero marginal cost, and full heterogeneity) can be combined, and the under-performance of the reward program carries through due to the same "down payment" argument. This accommodates, for example, a tiered reward program which is a combination of a BXGY program with credit-specific pricing for different units of Y. We also show that our main result is robust to "present bias" in the form of quasi-hyperbolic discounting.[2]

We then consider the profitability of reward programs under competition, where a switching costs argument may justify their use. We consider a Hotelling model with a continuum of customers having heterogeneous valuations and firms located at each end. Customer heterogeneity is as general as in the monopoly case. The firms decide simultaneously whether to offer a B1G1 program or charge spot prices. Again, the reward program underperforms relative to spot prices: the only Nash equilibrium is for both firms to offer spot prices. The intuition parallels that in the monopoly case. Regardless of whether its competitor offers a reward program or spot prices, the firm increases its demand and profits by offering spot prices due to the same "down-payment" argument. The reward program does increase switching costs – when a firm introduces a reward program, potential customers are more likely to buy from it if they hold a credit with the firm. However, it is not profitable for the firm to induce customers to earn these credits.

Finally, we consider a reward program with finite expiration of credits and show, in the monopoly case, that with such a feature the reward program may yield higher profits than spot prices. The introduction of expiring credits highlights an additional feature of reward programs in addition to time discounting. To illustrate, compare a B1G1 program to spot prices. Spot prices involve only period-by-period purchases, and the monopolist sells only to customers with current valuations above the price. The B1G1 program, in contrast, "bundles" a spot purchase with a future purchase. Since future valuations are uncertain, customers

[1] This increase in demand means that customers are also better off under spot prices.

[2] There are design features or settings that we do not consider. These include reward programs not controlled by the seller (e.g., a franchisee) (Chung et al., 2022), reward accumulation based on monetary rather than quantity purchases (Chun and Ovchinnikov, 2019), and coalitions of reward programs (Lederman, 2007).

will value the future product according to their expected rather than realized valuation. Since expected valuations are more similar across customers than realized, the monopolist's price-sales tradeoff is reduced, and it can charge a relatively higher price without losing commensurate sales. The logic is similar to the smoothing of valuations across different customer segments in the classical bundling literature (Adams and Yellen, 1976; Bakos and Brynjolfsson, 1999, 2000; Stremersch and Tellis, 2002) and models of advance purchasing (Shugan and Xie, 2000), in which a monopolist increases profits by pre-selling units, as the future expected value is more uniform than the later spot values. This "bundling" feature of reward programs unravels with no credit expiration because customers delay their future consumption until they realize a sufficiently high valuation.

The remainder of the paper is organized as follows. The next section reviews related studies. Section 3 presents the monopoly model, Section 4 the duopoly model, and Section 5 the finite-expiration case. Section 6 concludes.

2 Related Literature

Our work relates to three branches of existing literature on reward program profitability: theoretical rational models, theoretical behavioral models, and empirical analyses. Here we review each branch and identify our contribution. We then review other rationales in the literature for the profitability of reward programs.

2.1 Theoretical rational models

The earliest justifications for reward programs focused on switching costs. In a competitive environment, launching a reward program increases customers' stickiness to a firm and softens price competition (Kim et al., 2001; Singh et al., 2008). For tractability, these papers make simplifying assumptions, including a binary distribution of customer preferences and a two-period setting. In equilibrium, one or both firms launch a reward program. Two recent papers, Bazargan et al. (2018, 2020), extend the time horizon to multiple, but finite, periods. Assuming a logit distribution of customer valuations, these papers use numerical analysis to show the existence of different possible equilibria, including one in which neither firm launches a reward program. Relative to these studies, our work analytically examines a duopoly with infinitely-lived customers and a very general distribution of customer preferences and finds that neither firm offers a reward program in equilibrium. We also explain the necessity of finite expiration, implicitly assumed in earlier works with finite periods, to make the reward program profitable by comparing it to no expiration in an infinite-horizon setting. Besides relaxing assumptions for greater realism, we provide new insights into the switching costs argument. Although the reward program creates switching costs after customers make their initial purchases, it is unprofitable for firms to introduce the program ex-ante if credits do not expire and customers are fully forward-looking.

More recently, Sun and Zhang (2019) reveals another rationale, price discrimination, for the profitability of reward programs. In the paper, a monopolist can choose whether to launch a reward program with finite-credit expiration. Infinitely-lived customers are heterogeneous along two binary dimensions: product valuation and exogenous market participation frequency. In this setting, a reward program is more profitable than spot pricing when purchase frequency and valuations are negatively correlated. This is because the frequency reward program shrinks the difference in valuations between the two types of customers: the valuation for high-valued, low-participation customers is discounted as their credits are likely to expire, making them more similar to the low-valued but high-participation customers, whose valuations are enhanced by redeeming credits before expiration. Firms can extract more profit because they face more homogeneous customers.

We also identify finite-credit expiration as one critical reason for the profitability of reward programs but for a different reason. The reward program with finite expiration expands demand for future products as customers make decisions based on expected rather than realized payoffs. Such an effect does not rely on customer heterogeneity and exists in markets with homogeneous customers. Relatedly, we show the necessity of finite expiration by proving the sub-optimality of launching a reward program without expiration.[3] The finite-expiration result shares similar logic to models of advance purchases (Shugan and Xie, 2000; Xie and Shugan, 2001). These papers show that a monopolist can increase its profits by pre-selling units consumed later rather than selling at spot prices in the future. Similar to customer values for finite-expiration rewards, values for advance purchases are based on customers' expected future values, which are more uniform than future spot values. This allows the monopolist to sell to more customers, even though the advance price may be below the spot price that would be paid in the future by only higher-valuation customers. Alexandrov and Özlem Bedre-Defolie (2014) makes a similar analogy between advance purchases and the traditional bundling literature, as we do for finite-expiration reward programs. In contrast to our model, these papers consider two-period models in which customers purchase only in the second period, either at the spot price

[3] Liu et al. (2021) finds similar results in a different, more restrictive setting and using numerical analysis. Liu et al. (2021, p. 1840) states (using the acronym "BXGO" for a buy-X-get-one-free program) that, in the absence of these conditions: "We also conducted extensive numerical studies and did not find any parameter set under which the BXGO program improves the firm's profit."

or the previously-paid advance price.[4] We show that this intuition carries over to a fully dynamic model in which customers can qualify for a reward in any period.

Besides the new insights on switching costs and finite-expiration rewards, our model allows for more general customer heterogeneity and endogenous purchase frequencies and provides analytical results that guarantee robustness. We provide intuitions for these results and examine their generalizability along other reward-program dimensions, such as BXGY programs, credit-specific pricing, and time-inconsistent preferences.

There are a number of analytical papers that examine the design features of reward programs conditional on their being offered. These include reward timing (immediate- versus delayed-redemption), reward type (in-kind products, unrelated products, or price discounts), and rules for accumulating and redeeming points (Chun et al., 2020; Kopalle et al., 2012; Kim et al., 2021). These decisions depend on customer attributes, including heterogeneity of preferences across individuals and variation in preferences over time (Shin and Sudhir, 2010), rate of time preference, and frequency of shopping incidents (Sun and Zhang, 2019).

2.2 Theoretical behavioral models

Another branch of literature suggests that reward programs take advantage of customers' non-rational behaviors. These include reward effort to alleviate guilt (Kivetz and Simonson, 2002), reward effort as a perceived advantage over others (Kivetz and Simonson, 2003), unused credits in the form of "slippage" (Lu and Moorthy, 2007), different mental accounting for credits and cash (Zhang and Breugelmans, 2012; Stourm et al., 2015), increased purchases due to "points pressure" (Taylor and Neslin, 2005; Kivetz et al., 2006; Kopalle et al., 2012; Wang et al., 2016), "medium maximization" (Hsee et al., 2003), "rewarded-behavior" mechanisms (Taylor and Neslin, 2005; Drèze and Nunes, 2011; Kopalle et al., 2012), extra utility from point redemption (Rossi, 2018; Liu et al., 2021), the existence of hurdle costs for point redemption (Liu et al., 2021), and other bounded rationality effects (Liu and Ansari, 2020). We rule out behavioral elements to establish a benchmark of reward program profitability. The effects from any behavioral elements have to be large enough to compensate for the inferiority of the reward program under rationality.

[4]The finite-period setting is necessary for advance purchases to increase profits. As in our model, if customers could wait until their valuation is high in these models, the averaging of expected future values would unravel.

2.3 Empirical analyses

Empirical studies find mixed results on reward-program effects conditional on their being offered. Sharp and Sharp (1997) finds no evidence that a retailer reward program increases repeat purchases by customers, while Iyengar et al. (2022) finds a reward program increases purchases but not necessarily profits. Liu (2007) finds that a convenience-store reward program does not change the purchase behavior of ex-ante frequent buyers but does accelerate purchase frequency and size for less-frequent buyers. Lal and Bell (2003) finds that a grocery reward program is profitable because it reduces "cherry-picking" by infrequent customers, which more than compensates for rewards paid out to frequent customers. Heerde and Bijmolt (2005) decomposes the effect of different promotions on reward program members versus non-members and assesses the profitability of the promotions. Because they consider existing programs, these papers do not evaluate the incentive to introduce them relative to spot pricing. There are two empirical papers that examine the introduction of reward programs. Gopalakrishnan et al. (2021) finds that a hair salon reward program increases profits by reducing attrition. Rossi and Chintagunta (2022) finds prices increase in the later periods after a gas station adopts a reward program, consistent with "lock-in" due to switching costs.

There are a number of empirical papers that find evidence of switching costs conditional on a reward program being offered.[5]. These papers measure switching costs or purchase frequency at different levels of credit accumulation or nearness to achieving a higher status. Lewis (2004) finds that an online merchant reward program increases purchase frequencies, holding prices fixed. Orhun et al. (2022) finds that airline frequent flyer participants sacrifice utility when they are close to achieving a higher status relative to being far away. Hartmann and Viard (2008), on the other hand, find that the switching costs created by a golf reward program are small and apply only to infrequent customers who are near to earning a reward, a state rarely reached. In contrast to these papers, we ask whether it is profitable to introduce a reward program.

Finally, our paper relates to structural empirical models of switching costs and reward programs, (Lewis, 2004; Hartmann, 2006; Hartmann and Viard, 2008; Dubé et al., 2009; Kopalle et al., 2012) as our distributional assumptions accommodate these commonly-used models.

2.4 Other rationales

Our results imply that additional features are required to make reward programs profit-enhancing. Besides finite-expiration terms, the existing literature suggests a few reasons within the context of pure rationality.

[5]Papers examining switching costs independent of reward programs include Dubé et al. (2009), Goettler and Clay (2011), Cosguner et al. (2017) and Lei et al. (2024)

Shugan (2005) argues that reward programs, especially in travel, may take advantage of a principal-agent problem between decision-maker and payer. However, Basso et al. (2009) argues this may not hold in duopoly competition. Orhun et al. (2022) provide empirical evidence for this moral-hazard argument. Shugan (2005) offers other possible motivations for the use of reward programs, including price discrimination on redemption efforts, shifting revenues and costs, and implementing quantity discounts. Reward programs may also be employed to strategically manage limited capacity in order to soften price competition (Kim et al., 2004).

3 Monopoly Market

This section examines a monopolist choosing between a reward program and spot pricing. We begin by considering homogeneous customers with independent random utility across periods and describe customers' decisions under spot pricing and a B1G1 reward program. We then present our main result that, given a weak assumption about the distribution of random utility, the firm always chooses spot pricing. The underperformance of the reward program results from its effect on customer demand. We show that customers are better off, and therefore have higher demand, under spot pricing with price $p/2$ compared to a B1G1 program with price p. We describe the intuition behind this result. We then show that the underperformance of reward programs extends to much more general specifications of the reward program structure. These include a nonzero marginal cost, a buy-X-get-Y-free (BXGY) program, credit-specific pricing, very general specifications of customer heterogeneity, and a form of time-inconsistent customer preferences.

3.1 Model setup

We consider a monopolist facing a group of (ex-ante) homogeneous customers. In each period, each customer realizes a random utility $v \sim F(\cdot)$, independent and identically distributed across customers and over time. Let $G(v) = 1 - F(v)$ be the survival function of the distribution. Customers are infinitely-lived and maximize their lifetime benefit applying the discount factor β. This does not mean that customers literally live forever. Such a geometric discounting utility model is consistent with a finite but uncertain lifetime or "shopping lifespan" (Blanchard, 1985).[6] We relegate all proofs to Appendix A unless otherwise specified.

[6] Suppose that a customer has an expected lifespan of $\frac{1}{\rho}$ (probability ρ of death in each period with $0 < \rho < 1$). The customer's discount factor can be redefined as $\beta = \lambda(1-\rho)$ where λ is the time discount factor conditional on surviving to the next period. ρ need not necessarily refer to physical death but rather to a "shopping lifespan". For example, a customer may face some probability that they will no longer have need of the firm's services because they move or their circumstances change.

3.2 Spot pricing

If the monopolist charges a spot price of p per unit in each period, the per-period demand is:

$$D_S(p) = \Pr(v > p) = G(p). \tag{1}$$

3.3 B1G1 reward program

If the monopolist offers a B1G1 program with price p, customers' payoffs depend on whether they hold a credit ($s \in \{0, 1\}$). They pay price p to consume if $s = 0$, but pay nothing if $s = 1$. Their decision rule can be described by a Bellman equation. Let $u_a(v, s)$ be the payoff for a customer with realized random utility v holding credits s and who takes the action $a \in \{0, 1\}$ representing no-purchase and purchase:

$$\begin{aligned} u_1(v, s) &= v - p \cdot I(s = 0) + \beta \cdot w(1 - s) && (2) \\ u_0(v, s) &= \beta \cdot w(s), && (3) \end{aligned}$$

where $I(\cdot)$ is the indicator function and $w(s)$ is the continuation value in the next period as a function of the state variable:

$$\begin{aligned} w(0) &= E \max \left(u_1(v, 0), u_0(v, 0)\right) && (4) \\ w(1) &= E \max \left(u_1(v, 1), u_0(v, 1)\right). && (5) \end{aligned}$$

For any values of $\{G(\cdot), \beta, p\}$, the value functions can be solved for using the above two equations.

Given the value functions, and therefore utility, customers' consumption probabilities under the two states ($q_s, s \in \{0, 1\}$) are:

$$\begin{aligned} q_0 &= \Pr\left(u_1(v, 0) > u_0(v, 0)\right) = G(p - \beta \Delta w) && (6) \\ q_1 &= \Pr\left(u_1(v, 1) > u_0(v, 1)\right) = G(\beta \Delta w), && (7) \end{aligned}$$

where $\Delta w = w_1 - w_0$. Only customers not holding a credit ($s = 0$) will pay when consuming, so we solve for the steady-state probability of both states. Let (r_0, r_1) be the probability of customers holding zero and one credit respectively with $r_0 + r_1 = 1$. In a steady-state:

$$r_0 = q_1 r_1 + (1 - q_0) r_0, \tag{8}$$

which gives

$$r_0 = \frac{q_1}{q_1 + q_0}. \tag{9}$$

Thus, demand is:

$$D_R(p) = q_0 r_0 = \frac{q_0 q_1}{q_0 + q_1} = \left(\frac{1}{q_0} + \frac{1}{q_1}\right)^{-1}. \tag{10}$$

3.4 Comparing B1G1 program and spot pricing

Two observations about the demand for the reward program allow a precise comparison of it to spot pricing without the necessity of analytical solutions. First, the demand function in Equation (10) is a harmonic average of the purchase likelihood in two states: q_0 and q_1. Second, the average of the arguments of these likelihoods, $p - \beta\Delta w$ and $\beta\Delta w$ shown in Equations (6) and (7), is $p/2$, which conveniently does not depend on the complicated parts of the value functions. At the same time, the spot pricing demand depends only on price p. This makes it possible to impose a weak convexity condition and apply Jensen's inequality to compare the two demands:

Assumption 1. *$H(v) = (G(v))^{-1}$ is convex.*

This assumption is met by most commonly-used distributions (as shown in Appendix B) including those used in empirical structural estimation. It is implied by other commonly-used assumptions, such as an increasing hazard rate and the monotone likelihood ratio property (Milgrom, 1981; Mayzlin and Shin, 2011; Miklós-Thal and Zhang, 2013; Lei et al., 2024). Given this assumption, our main result is:

Proposition 1. *Under Assumption 1,*

$$D_R(p) \leq \frac{1}{2} D_S\left(\frac{p}{2}\right), \tag{11}$$

where $D_S(p)$ and $D_R(p)$ are defined in Equations (1) through (10).

Proof. Since $H(v)$ is convex, Jensen's inequality implies:

$$\frac{1}{2}\left(H(p - \beta\Delta w) + H(\beta\Delta w)\right) \geq H\left(\frac{p}{2}\right). \tag{12}$$

Plug in to get:

$$\left(\frac{1}{q_0} + \frac{1}{q_1}\right) \geq \frac{2}{G\left(\frac{p}{2}\right)}. \tag{13}$$

Taking the reciprocal of both sides yields Equation (11). □

We can rearrange Equation (11) as $2 \cdot D_R(p) \leq D_S(p/2)$ to interpret it. For a B1G1 program with price p, two units ($2 \cdot D_R(p)$) are purchased per customer – one consumed now and one consumed later. On the other hand, under spot pricing with price $p/2$, one unit ($D_S(p/2)$) is purchased per customer and consumed now. The result indicates that customers facing the two pricing schemes with identical per-unit prices will consume weakly more under spot pricing. This is because, under the reward program, customers have to make a down payment for the unit consumed in the future. This makes them strictly worse off than if they were allowed to pay and consume as they go.

A direct implication of this result is that the monopolist's profits are lower under the reward program. Let (π_R^*, π_S^*) be the optimal profits under the reward program and spot pricing respectively. Then:

Corollary 1. *Under Assumption 1, applying the result in Equation* (11)*:*

$$\pi_R^* = \max_p p \cdot D_R(p) \leq \max_p \frac{p}{2} \cdot D_S(\frac{p}{2}) = \pi_S^*. \tag{14}$$

This corollary follows directly from multiplying both sides of Equation (11) by price p. It implies that profits are greater under spot pricing for *any* price; therefore, profits must be greater under the optimal price. A firm charging price p under a B1G1 reward program can always earn higher profits by charging $p/2$ under spot pricing.

3.5 Extensions

Our main result in Proposition 1 is quite robust and can be extended along several dimensions, including a non-zero marginal cost, a BXGY program, credit-specific pricing, any type of preference heterogeneity, and a form of time-inconsistent customer preferences. Any of these extensions can be combined with each other. For example, a reward program with tiers would combine a BXGY program with credit-specific pricing and potentially non-zero marginal cost. In fact, all these elements can be present in a single model and the results hold.

3.5.1 Non-zero marginal cost

In this subsection, we relax the assumption of a zero marginal cost. Assume that the monopolist incurs a constant marginal cost, c, to serve a unit of product. This occurs at the time of consumption rather than purchase (e.g., in a coffee reward program at the time the coffee is served and drunk). The demand function in Equation (11) does not change, but the profit functions now include the cost of serving. Let $\pi_S(p)$ and $\pi_R(p)$ be the profit functions for spot pricing and the reward program respectively:

$$\pi_S(p) = (p - c) \cdot D_S(p) \tag{15}$$

$$\pi_R(p) = p \cdot D_R(p) - c \cdot (q_0 r_0 + q_1 r_1). \tag{16}$$

It is easy to derive from Equation (9) that $q_0 r_0 = q_1 r_1 = D_R(p)$ so that:

$$\pi_R(p) = (p - 2c) \cdot D_R(p). \tag{17}$$

Applying a similar argument as in Equation (14), we have:

$$\pi_R^* = \max_p (p - 2c) \cdot D_R(p) \leq \max_p (p/2 - c) \cdot D_S(\frac{p}{2}) = \pi_S^*. \tag{18}$$

3.5.2 BXGY program

A natural extension of the main result is a more general structure of earning and redeeming credits: a BXGY program for integers $X, Y \in \{1, 2, 3, ...\}$. The "down-payment" intuition holds in this more general case.

Consider a BXGY program with price p. State variables can now take the values $s \in \{0, 1, ..., X, X + 1, ..., X + Y - 1\}$, which cycle back to zero after consumption of $X + Y$ unites. For $s < X$, customers have to pay for the next consumption; whereas for $s \geq X$, customers redeem credits for free consumption. The payoff functions are:

$$u_1(v, s) = v - p \cdot I(s < X) + \beta \cdot w(g(s)) \tag{19}$$

$$u_0(v, s) = \beta \cdot w(s), \tag{20}$$

where the state transition when the customer consumes is $g(s) = s + 1$ if $s < X + Y - 1$, and $g(s) = 0$ if

$s = X + Y - 1$. The value function is defined similarly as:

$$w(s) = E \max \left(u_1(v, s), u_0(v, s) \right). \tag{21}$$

The choice probabilities (q_s) and steady-state distribution (r_s) are:

$$\begin{aligned} q_s &= \Pr\left(u_1(v, s) > u_0(v, s) \right) = G\left(p \cdot I(s < X) - \beta \Delta w(s) \right) & (22) \\ r_{(g(s))} &= r_s \cdot q_s + r_{g(s)} \cdot (1 - q_{g(s)}), & (23) \end{aligned}$$

where $\Delta w(s) = w\left(g(s)\right) - w(s)$, and $\sum_{s=0}^{X} \Delta w(s) = 0$. Only customers with $s < X$ will pay, so demand is:

$$D_{RX}(p) = \sum_{s=0}^{X-1} r_s q_s, \tag{24}$$

and we have a result analogous to that of the B1G1 program:

Proposition 2. *Under Assumption 1, the BXGY reward program yields lower profits than spot pricing with the same per-unit price:*

$$D_{RX}(p) \leq \frac{X}{X+Y} D_S \left(\frac{X}{X+Y} \cdot p \right). \tag{25}$$

3.5.3 Credit-specific pricing

The baseline model can be extended to allow for prices that vary by credit (p_s). For example, a buy-one-get-one at 50% off program. The standard B1G1 program is a special case where the price is p for $s = 0$ and 0 for $s = 1$. The reward program's underperformance relative to spot pricing carriers over to this extension:

Proposition 3. *Under Assumption 1, a B1G1 program with credit-specific pricing* (p_0, p_1) *yields lower profits than spot pricing at price* $\bar{p} = \frac{1}{2}(p_0 + p_1)$.

3.5.4 Heterogeneity

A potential benefit of a reward program is price discrimination. It charges different effective prices to customers with different purchase frequencies: frequent customers experience a lower price than infrequent ones. Such an argument does not appear to be valid when purchase frequencies are endogenously determined. We show this by extending our model to a rich specification of customer heterogeneity where customers' types are captured by one stochastic dimension, v, and show that our main result still holds.

Consider a finite mixture of the type distribution in which there are K types of customers parameterized by $\{(G^k(\cdot), \beta^k)\}_{k=1}^K$, each with probability λ^k and $\sum_k \lambda^k = 1$. Applying Section 3.3 separately to each type, the demand for each type is $\{D_R^k(p), D_S^k(p)\}$, and the result in (11) holds for each type:

$$D_R^k(p) \leq \frac{1}{2} D_S^k(\frac{p}{2}). \tag{26}$$

Therefore, the reward program yields lower profits than spot pricing:

$$\pi_R^* = \max_p p \cdot \sum_k \lambda^k D_R^k(p) \leq \max_p \frac{p}{2} \sum_k \lambda^k D_S^k(\frac{p}{2}) = \pi_S^*. \tag{27}$$

3.5.5 Quasi-hyperbolic discounting

Time-inconsistent preferences and lack of self-control are sometimes used to explain agents' intertemporal behaviors (Laibson, 1997; O'Donoghue and Rabin, 1999; Jain, 2012; Jain and Li, 2018; Amaldoss and Harutyunyan, 2023). We examine the profitability of a reward program when customers are time-inconsistent and exhibit quasi-hyperbolic discounting. Consider naive customers who think their behavioral pattern in the future is determined by time-consistent preferences with discount factor β, whereas in the current period, their discount factor is smaller ($\gamma \cdot \beta$ with $\gamma < 1$). Suppose the firm launches a B1G1 program at a price p. The customer thinks his value function in the future, denoted as $w(s;\beta)$, is purely rational, determined by (2) – (5). In the current period, however, their utility is:

$$\begin{aligned} \tilde{u}_1(v,s) &= v - p \cdot I(s=0) + \gamma\beta \cdot w(1-s;\beta) && (28) \\ \tilde{u}_0(v,s) &= \gamma\beta \cdot w(s;\beta). && (29) \end{aligned}$$

Their purchase likelihood is:

$$\begin{aligned} \tilde{q}_0 &= \Pr\left(\tilde{u}_1(v,0) > \tilde{u}_0(v,0)\right) = G(p - \gamma\beta \cdot \Delta w(\beta)) && (30) \\ \tilde{q}_1 &= \Pr\left(\tilde{u}_1(v,1) > \tilde{u}_0(v,1)\right) = G(\gamma\beta \cdot \Delta w(\beta)), && (31) \end{aligned}$$

and the steady-state distribution $(\tilde{r}_0, \tilde{r}_1)$ is determined as in (9). The demand is:

$$\tilde{D}_R(p) = \tilde{q}_0 \tilde{r}_0 = \frac{\tilde{q}_0 \tilde{q}_1}{\tilde{q}_0 + \tilde{q}_1} = \left(\frac{1}{\tilde{q}_0} + \frac{1}{\tilde{q}_1}\right)^{-1}. \tag{32}$$

The demand from charging a spot price of $p/2$ is the same as in (1). The result from Proposition 1 still holds here:

$$\tilde{D}_R(p) \leq \frac{1}{2} D_S \left(\frac{p}{2} \right). \tag{33}$$

Even when facing behavioral customers exhibiting time-inconsistent preferences, it is not optimal for the firm to launch a reward program.

4 Duopoly Market

Competition is a prevalent reason given to explain the existence of reward programs. The argument is that after inducing customers to purchase, a firm's reward (or nearness to it) creates switching costs which makes it less likely they purchase from competitors. As a result, the firm can exercise market power. However, these models assume that the firm has already established the program. A reward program may increase switching costs but still not be worth establishing. In particular, the alternative to a reward program is spot pricing. In this section, we examine the overall profitability of launching a reward program relative to spot pricing under competition by extending the monopoly model to a duopoly in which the firms simultaneously choose whether to offer a reward program or spot pricing. We prove that the only Nash Equilibrium is for both firms to charge spot prices. The intuition parallels the monopoly analysis – either firm can realize more demand under spot pricing than under a reward program regardless of its competitor's choice. We first introduce the setup and then solve for the focal firm's best-response to the other firm choosing either spot pricing or a reward program.

4.1 Model setup

Two firms (denoted 0 and 1) are located at the extremes of a Hotelling model of length one and face customers whose preferences are both horizontally and vertically differentiated. Each customer in a period is denoted by (v, t) where t is their time-persistent location parameter and v is their random utility, which changes between periods with survival function $G(\cdot)$. A customer with parameter t incurs a transportation cost t when buying from firm 0 and $(1 - t)$ when buying from firm 1. The results hold for any distribution of t on the unit interval and with the same general preference distribution for v assumed in the monopoly model. For simplicity, we assume that both firms choose between offering a B1G1 reward program and spot pricing. The timing of the model is:

1. Both firms ($j \in \{0,1\}$) choose (p_j, x_j) simultaneously, where p_j is price and $x_j \in \{R, S\}$ is the pricing structure: reward program (R) or spot pricing (S). This choice remains over all periods.

2. Given (p_j, x_j), customers take one of three actions in each period: $a \in \{0, 1, \phi\}$, where $a = j$ means the customer chooses firm-j's product and $a = \phi$ means the customer does not consume.

The remainder of this section proves the following result:

Proposition 4. *The only Nash equilibrium for both firms is:*

$$x_0 = x_1 = S \tag{34}$$

$$p_0 = p_1 = p^*, \tag{35}$$

where p^* is the equilibrium price in a standard Hotelling model.

We assign firm 0 as the focal firm and derive its best-response in two cases: (1) when firm 1 chooses spot pricing, and (2) when firm 1 chooses a reward program. For both cases, we show that our demand comparison result in Proposition 1 holds. That is, firm 0's best response is always to choose spot pricing.

4.2 Firm 1 chooses spot pricing ($x_1 = S$)

We first solve for firm 0's profits when launching a reward program and spot pricing, respectively, given that firm 1 chooses spot pricing at price p_1. We then prove that firm 0's profits are higher with spot pricing in both cases, regardless of p_1.

4.2.1 Case 1: firm 0 chooses reward program ($x_0 = R$)

If firm 0 chooses a reward program, customers have one state variable (firm 0 credits) in making their purchase decisions. For customers of type t the value functions $\left(w = (w_0, w_1)^T\right)$, corresponding to holding or not holding a credit, are (going forward we will sometimes suppress t for simplicity):

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = E \max \begin{pmatrix} v - t - p + \beta w_1, & v - (1-t) - p_1 + \beta w_0, & \beta w_0 \\ v - t + \beta w_0, & v - (1-t) - p_1 + \beta w_1, & \beta w_1 \end{pmatrix}. \tag{36}$$

Define:

$$\Delta w = w_1 - w_0. \tag{37}$$

We have the following properties:

Proposition 5. *For any t, the value function solved by Equation (36) satisfies:*

(a) $w_0(t) \leq w_1(t)$

(b) $\beta \Delta w(t) \leq \frac{p}{2}$.

We now derive the consumption probability under the two states (q_0, q_1) and the corresponding demand functions for customers located at t. For customers with $s = 0$, they will buy from firm 0 if both:

$$v - (p + t) + \beta w_1 \geq \beta w_0, \text{ and} \tag{38}$$

$$v - (p + t) + \beta w_1 \geq v - (p_1 + 1 - t) + \beta w_0. \tag{39}$$

This implies the choice probability is:

$$q_0(t) \cdot C_0(t) = G\left(p + t - \beta \Delta w(t)\right) \cdot I\left(p - \beta \Delta w(t) \leq p_1 + 1 - 2t\right), \tag{40}$$

where $C_0(t)$ is an indicator variable denoting market coverage of firm 0 at location t for customers who do not hold a credit and $I(\cdot)$ is the indicator function. For $s = 1$, a customer with type t will buy from firm 0 if both:

$$v - t + \beta w_0 \geq \beta w_1, \text{ and} \tag{41}$$

$$v - t + \beta w_0 \geq v - (p_1 + 1 - t) + \beta w_1, \tag{42}$$

which implies:

$$q_1(t) \cdot C_1(t) = G\left(t + \beta \Delta w(t)\right) \cdot I\left(\beta \Delta w(t) \leq p_1 + 1 - 2t\right), \tag{43}$$

where $C_1(t)$ is an indicator variable denoting market coverage of firm 0 at location t for customers who hold a credit.

Depending on their realization of v, customers located at t not holding a credit ($s = 0$) choose between firm 0 and not consuming ($a \in \{0, \phi\}$) if $C_0(t) = 1$, or choose between firm 1 and not consuming ($a \in \{1, \phi\}$) if $C_0(t) = 0$. Applying the result in Proposition 5 in comparing Equations (40) and (43), their choice between firms, as determined by t, follows:

$$C_0(t) \leq C_1(t). \tag{44}$$

In other words, customers are more likely to choose firm 0 when they have a credit. This reflects the switching costs that previous papers have analyzed (Lewis, 2004; Orhun et al., 2022; Hartmann and Viard, 2008). However, this presupposes the existence of a reward program rather than whether it is optimal to introduce the program.

What remains is mechanical. We follow the monopoly case (analogous to Equation (10)) and derive the steady-state distribution and the demand function for firm 0. In the steady state, firm 0 only sells to customers with $C_0(t) = 1$.[7] The demand function is therefore:

$$D_{RS}(p,t) = \frac{q_0(t)q_1(t)}{q_0(t)+q_1(t)} \cdot I(p - \beta\Delta w(p,t) \leq p_1 + 1 - 2t), \tag{45}$$

where the first and second subscripts of D denote firm 0's and firm 1's choices, respectively, of spot pricing (S) versus a reward program (R).

4.2.2 Case 2: firm 0 chooses spot pricing ($x_0 = S$)

When firm 0 chooses spot pricing at price p, the model collapses to the standard Hotelling model. A customer with (v,t) will buy from firm 0 if:

$$v - (p+t) \geq 0, \text{ and} \tag{46}$$

$$v - (p+t) \geq v - (p_1 + (1-t)), \tag{47}$$

so that the demand from customers located at t is:

$$D_{SS}(p,t) = G(p+t) \cdot I(p + t \leq p_1 + (1-t)). \tag{48}$$

4.2.3 Firm 0's best response comparing cases 1 and 2

The demand functions in both cases are similar to those in the monopoly setting, but multiplied by a dummy variable which depends on t. It is straightforward to apply Proposition 5 to show that the relationship in the monopoly case of Proposition 1 also holds here:

[7] Since $C_0(t) \leq C_1(t)$ (the focal firm's market coverage for customers with a credit extends further on the Hotelling line than does the coverage for customers without a credit), there is an interval in which $C_0(t) = 0$ but $C_1(t) = 1$. These are not customers of firm 0 in the steady state because even if the customers in this interval are endowed with a credit in the initial period, once they redeem the credit they do not return to this state again.

Proposition 6. *When $x_1 = S$, comparing the demands for firm 0's two possible choices, $D_{RS}(p)$ and $\frac{1}{2}D_{SS}(\frac{p}{2})$ specified by Equations (45) and (48), we have:*

(a) Applying Proposition 5, firm 0 can sell to more customers under a spot pricing than a reward program:

$$I\left(p - \beta\Delta w(p,t) \leq p_1 + 1 - 2t\right) \leq I\left(\frac{p}{2} + t \leq p_1 + (1-t)\right). \tag{49}$$

(b) This implies, by Proposition 1:

$$D_{RS}(p,t) \leq \frac{1}{2}D_{SS}\left(\frac{p}{2}, t\right). \tag{50}$$

(c) Therefore, firm 0 can achieve higher profits with spot pricing at price $p/2$ than with a reward program at price p.

4.3 Firm 1 launches a reward program ($x_1 = R$)

We first solve for firm 0's profit when offering a reward program versus offering spot pricing, respectively, given that firm 1 chooses a reward program at price p_1. We then prove that firm 0's profits are higher with spot pricing in both cases, regardless of p_1.

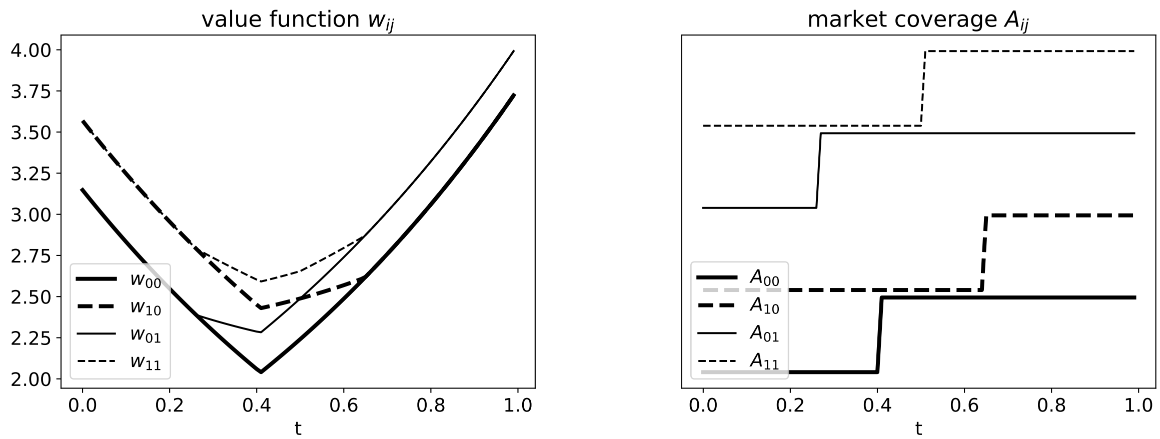
4.3.1 Case 3: firm 0 chooses a reward program ($x_0 = R$)

When both firms choose a reward program, each customer has two state variables $(s_0, s_1) \in \{0,1\}^2$, representing the credits they hold from firm 0 and firm 1 respectively. The value functions w_{jk} given state variables $j = s_0$ and $k = s_1$ collected as $w = (w_{00}, w_{10}, w_{01}, w_{11})^T$, for customers located at t, are:

$$\begin{pmatrix} w_{00} \\ w_{10} \\ w_{01} \\ w_{11} \end{pmatrix} = E\max\begin{pmatrix} v-(p+t)+\beta w_{10}, & v-(p_1+1-t)+\beta w_{01}, & \beta w_{00} \\ v-t+\beta w_{00}, & v-(p_1+1-t)+\beta w_{11}, & \beta w_{10} \\ v-(p+t)+\beta w_{11}, & v-(1-t)+\beta w_{00}, & \beta w_{01} \\ v-t+\beta w_{01}, & v-(1-t)+\beta w_{10}, & \beta w_{11} \end{pmatrix}. \tag{51}$$

Using similar arguments as in the previous subsection, for customers located at t their realization of v determines a choice between buying or not rather than a choice between firms (the location t, on the other hand, influences their choice between firms). We use $A_{jk}(t) \in \{0,1\}$ to indicate the choice of firm 0 or 1 for customers located at t (if not choosing no purchase ϕ), given their state values $j = s_0$ and $k = s_1$.

Figure 1: Numerical examples of possible choices in duopoly model



Notes: The left figure shows value functions of customers at different states for the given parameter values and the right figure shows their choices. Parameter values are: $(p, p_1, \beta) = (0.8, 0.5, 0.8)$, and $v \sim$ Exponential(1).

Intuitively, if customers located at t choose the focal firm when they do not hold a credit ($s_0 = 0$), they will also do so when they hold a credit with the focal firm ($s_0 = 1$). That is, $A_{0k} = 0 \Rightarrow A_{1k} = 0$. Similarly, holding a credit for firm 1 makes customers less likely to choose firm 0 (i.e., $A_{j0} = 1 \Rightarrow A_{j1} = 1$). These two implications reflect the switching costs created by reward programs that previous papers have analyzed (conditional on firms offering a reward program). Figure 1 provides an example of this. It shows the value functions (w_{jk}) and the corresponding firm choices (A_{jk}) as a function of the location parameter t. Consistent with the switching costs argument, the figure shows $w_{10} > w_{00}$ and $w_{11} > w_{01}$. We formalize this argument in the following result:

Proposition 7. *Let w be the value functions solving (51), and define:*

$$A_{00} = I(-(p+t) + \beta w_{10} \leq -(p_1 + 1 - t) + \beta w_{01}) \tag{52}$$

$$A_{10} = I(-t + \beta w_{00} \leq -(p_1 + 1 - t) + \beta w_{11}) \tag{53}$$

$$A_{01} = I(-(p+t) + \beta w_{11} \leq -(1 - t) + \beta w_{00}) \tag{54}$$

$$A_{11} = I(-t + \beta w_{01} \leq -(1 - t) + \beta w_{10}). \tag{55}$$

Then:

$$A_{10} \leq \{A_{00}, A_{11}\} \leq A_{01}. \tag{56}$$

The market coverage for firm 0's sales is $\{t : A_{00} = 0\}$ – customers holding no credits with firm 0 that choose firm 0. Applying the above results, these same customers will choose firm 0 if they hold a credit with the firm ($A_{10} = 0$). Similarly, firm 1's market coverage for sales is $\{t : A_{00} = 1\}$ and customers who choose firm 1 when holding no credits will also choose firm 1 when they do ($A_{01} = 1$). This again illustrates the presence of switching costs (ex-post segmentation) conditional on a reward program being offered. That is, customers are less likely to purchase from the other firm if they hold a credit with a firm.

We show that as t becomes larger, firm 0 becomes less attractive. Such a monotonic property guarantees firm 0's market coverage $\{t : A_{00} = 0\}$ is an interval. Formally:

Proposition 8. *$A_{00}(p, p_1, t)$ increases in t.*

Firm 0's market coverage is therefore:

$$C_{00}^{RR}(p, t) = I(t \leq \bar{t}), \tag{57}$$

where $\bar{t}$ is determined by the following identity:

$$-(p + \bar{t}) + \beta w_{10}(\bar{t}) = -(p_1 + 1 - \bar{t}) + \beta w_{01}(\bar{t}). \tag{58}$$

The demand function is:

$$D_{RR}(p, t) = \frac{q_1(t)q_0(t)}{q_1(t) + q_0(t)} \cdot C_{00}^{RR}(p, t), \tag{59}$$

where $q_0(t)$ and $q_1(t)$ are defined in Equations (40) and (43) respectively.

4.3.2 Case 4: firm 0 chooses spot pricing ($x_0 = S$)

If firm 0 chooses spot pricing, customers have only one state variable, the credit balance with firm 1. The value functions are:

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = E \max \begin{pmatrix} v - (p + t) + \beta u_0, & v - (p_1 + 1 - t) + \beta u_1, & \beta u_0 \\ v - (p + t) + \beta u_1, & v - (1 - t) + \beta u_0, & \beta u_1 \end{pmatrix}. \tag{60}$$

Let $A_{s_1}^{SR}$ indicate the customers' choice of firm 1 with state variable s_1:

$$A_0^{SR} = I(-(p + t) + \beta u_0 \leq -(p_1 + 1 - t) + \beta u_1) \tag{61}$$

$$A_1^{SR} = I(-(p + t) + \beta u_1 \leq -(1 - t) + \beta u_0). \tag{62}$$

Then we can show:

Proposition 9. *Customers' choice probabilities have the following properties:*

(a) $A_0^{SR} \leq A_1^{SR}$

(b) $A_0^{SR}(p, p_1, t)$ *increases in* t.

Given this, the demand curve is:

$$D_{SR}(p,t) = G(p+t) \cdot C_0^{SR}(p,t), \tag{63}$$

where

$$C_0^{SR}(p,t) = I(-(p+t) + \beta u_0 \geq -(p_1 + 1 - t) + \beta u_1). \tag{64}$$

4.3.3 Firm 0's best response comparing cases 3 and 4

We now compare firm 0's demand when it launches a reward program ($D_{RR}(p)$) with price p, defined by Equation (59), to that when it offers spot pricing at half the price $\left(\frac{1}{2}D_{SR}\left(\frac{p}{2}\right)\right)$, defined by Equation (63). We proceed in two parts corresponding to the two components of demand in these equations: the demand from type t conditional on it choosing firm 0, and the probability that type t chooses firm 0. Using the same logic as in the monopoly case, we show that (conditional) demand by type t is less under the reward program with price p than it is under spot pricing with a price $\frac{p}{2}$. We then show that firm 0's market coverage is weakly smaller when launching a reward program than when it offers spot pricing at half the price (the choice probability in Equation (59) evaluated at p compared to the choice probability in Equation (63) evaluated at $\frac{p}{2}$):

$$C_0^{RR}(p,t) \leq C_{00}^{SR}\left(\frac{p}{2}, t\right). \tag{65}$$

Since the market coverages on both sides of this equation are decreasing in t and $C_{00}^{RR}(p, \bar{t}) = 1$, we only need to show that $C_0^{SR}\left(\frac{p}{2}, \bar{t}\right) = 1$. This would imply that the above identity holds for $t \leq \bar{t}$. This means that all customers who purchase from firm 0 when it offers a reward program, defined by the cutoff type ($\bar{t}$ in Equation (58)), will also choose firm 0 if firm 0 chooses spot pricing at half the price. Specifically:

Proposition 10. *When* $x_1 = R$*, comparing firm 0's demand under the two market structures* $D_{RR}(p)$ *and* $\frac{1}{2}D_{SR}(\frac{p}{2})$*:*

(a) Market coverage when firm 0 offers spot pricing at $\frac{p}{2}$ is weakly larger than when it launches a reward program at price p. That is, for $\bar{t}$ given by Equation (58):

$$C_0^{SR}(\frac{p}{2}, \bar{t}) = 1. \tag{66}$$

(b) This implies:

$$D_{RR}(p, t) \leq \frac{1}{2} D_{SR}\left(\frac{p}{2}, t\right). \tag{67}$$

(c) And firm 0 realizes higher profit when choosing spot pricing with price $p/2$ than when launching a reward program with price p.

5 Finite Expiration

This section analyzes a design feature of reward programs, finite-credit expiration, that can increase firm profits relative to spot pricing. This has been shown before in a simulation with a binary distribution of customer preferences (Liu et al., 2021), and analytically in a model with two dimensions of customer heterogeneity (Sun and Zhang, 2019).[8] We show, analytically, that a finite expiration reward program can outperform spot pricing in a different setting with only one-dimensional heterogeneity, due to a different underlying mechanism. Comparing our finite- and no-expiration results provides the intuition for why this is the case. To simplify the analysis, we examine the monopoly case with homogeneous customers and focus on an exponential distribution for v:

$$F(v) = \begin{cases} 1 - \exp(-\lambda v) & \text{if } v \geq 0 \\ 0 & \text{if else.} \end{cases} \tag{68}$$

The inverse of the survival function, $H(v) = \left(\frac{1}{1-F(v)}\right)^{-1}$, is globally convex satisfying Assumption 1. We consider a B1G1 program with one-period expiration: a credit for a free product must be consumed in the next period. We first derive the demand and profits for the reward program with expiring credits, and then show that the reward program can outperform spot pricing if the discount factor is large enough.

[8] An earlier paper (Chen et al., 2005) shows a similar result in a two-period model which imposes finite expiration by default.

5.1 Reward program with expiring credits

We follow a similar procedure as before: first describing the value functions, then the choice probabilities, then the steady-state distribution, and finally demand.

Let $u_a(v, s)$ be the value of taking action $a \in \{0, 1\}$ with $s \in \{0, 1\}$ credits. When $s = 0$:

$$u_1(v, 0) = v - p + \beta \cdot w(1) \tag{69}$$

$$u_0(v, 0) = \beta \cdot w(0), \tag{70}$$

which is identical to the case without expiring credits. However, when $s = 1$, the utility becomes:

$$u_1(v, 1) = v + \beta \cdot w(0) \tag{71}$$

$$u_0(v, 1) = \beta \cdot w(0), \tag{72}$$

where the value of not taking action ($u_0(v, 1)$), differs from the no-expiration case – the credit expires and the state becomes zero. The value functions are determined by the following system of equations:

$$w(1) = E \max(v + \beta w(0), \beta w(0)) = E(v) + \beta w(0) \tag{73}$$

$$w(0) = E \max(v - p + \beta w(1), \beta w(0)) = E \max(v - p + \beta \Delta w, 0) + \beta w(0), \tag{74}$$

where $\Delta w = w\,(1) - w\,(0)$. The first equation reflects the fact that customers will always redeem their credit since their valuation is always positive under the exponential distribution. Taking the difference between the two, Δw is the solution of:

$$\Delta w = E(v) - E \max(v - p + \beta \Delta w, 0). \tag{75}$$

Next, we solve for the consumption likelihood. When $s = 1$, customers will consume for sure since $v > 0$, and the credit will expire if not used:

$$q_1 = 1 = G(0). \tag{76}$$

When $s = 0$, the purchase likelihood is:

$$q_0 = \Pr(v - p + \beta w(1) \geq \beta w(0)) = G(p - \beta \Delta w). \tag{77}$$

The steady-state distribution of credits (r_0, r_1) is determined by:

$$r_1 = r_0 q_0 \tag{78}$$

$$r_0 + r_1 = 1, \tag{79}$$

which implies:

$$r_0 = \frac{1}{1+q_0}. \tag{80}$$

So the demand under a reward program with one-period expiration is:

$$D_R^F(p) = r_0 q_0 = \frac{q_0}{1+q_0} = \left(\frac{1}{q_0} + \frac{1}{q_1}\right)^{-1}, \tag{81}$$

where the last equality holds as $q_1 = 1$.

5.2 Comparing to spot pricing

Note that our main result in Proposition 1 does not hold in the current context. Although the demand $D_R^F(\cdot)$ is still a harmonic mean of the two purchase likelihoods q_0, and q_1; the quantity $q_1 = 1$ is higher than in the no-expiration case leading to the possibility that demand is higher here than under spot pricing. Specifically, applying the convexity prescribed in Assumption 1, we have:

$$D_R^F(p) \leq \frac{1}{2} G\left(\frac{1}{2}(p - \beta \Delta w)\right). \tag{82}$$

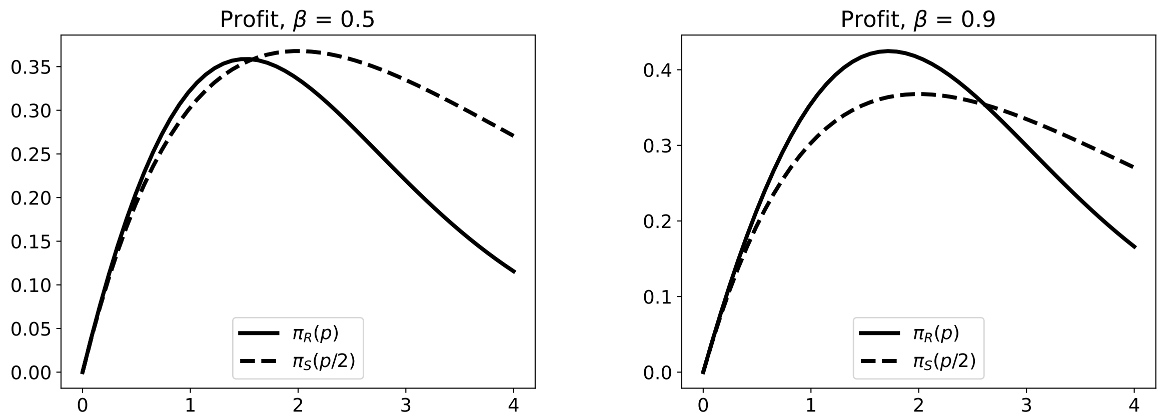
The right-hand-side may be larger than demand under spot pricing with price $\frac{p}{2}$: $1/2 \cdot D_S(p/2) = 1/2 \cdot G(p/2)$. Thus, demand with a finite-expiration program may exceed that under spot pricing.

Before deriving a more general result, we first compare profits in a numerical example with $\lambda = 1$ and two different discount factors: $\beta \in \{0.5, 0.9\}$. Without expiring credits, profits under spot pricing would always weakly exceed those under a reward program regardless of p since:

$$\pi_R(p) \leq \pi_S(\frac{p}{2}). \tag{83}$$

However, this does not necessarily hold when credits expire. Figure 2 compares profits under the reward program with expiring credits versus that under spot pricing: the left panel for $\beta = 0.5$ and the right for $\beta = 0.9$. As the figure demonstrates, reward program profits exceed those under spot pricing for small values

Figure 2: Profits under reward program with expiring credits versus spot pricing



Notes: The two panels compare profits under a reward program with finite (one-period) expiration of credits and price p $(\pi_R(p))$ to that under spot pricing with price $\frac{p}{2}$ $\left(\pi_S\left(\frac{p}{2}\right)\right)$ using an exponential distribution of preferences with $\lambda = 1$. The left panel assumes $\beta = 0.5$ and the right panel $\beta = 0.9$.

of p in both panels. However, at higher prices, profits under spot pricing exceed those under the reward program. The two profit functions have a single crossing point, and whether reward program profits exceed those under spot pricing depends on the position of the crossing point relative to the point of maximum profits. When $\beta = 0.5$, the crossing point occurs where spot program profits are still increasing and spot pricing outperforms the reward program. When $\beta = 0.9$, on the other hand, the crossing point occurs where spot profits are decreasing and the reward program outperforms spot pricing.

More generally, we can prove the following result:

Proposition 11. *For any value of λ, there exists $\bar{\beta}$, such that for all $\beta > \bar{\beta}$:*

$$\max_p p \cdot D_R^F(p) > \max_p p \cdot D_S(p). \tag{84}$$

Although we are not the first to propose that finite-expiration credits can lead to reward programs outperforming spot pricing in a non-behavioral setting, our explanation differs from previous studies (Sun and Zhang, 2019; Liu et al., 2021). The reward program with immediate expiration can be regarded as a bundle of two products: one "spot" product consumed today and one "future" product consumed tomorrow. If the firm charges separately for the two products, the "spot" price will maximize:

$$\pi_{sp}^* = \max_p p \cdot D(p) = \max_p p \cdot \exp(-\lambda p), \tag{85}$$

yielding optimal price $p^*_{sp} = 1/\lambda$ and profit $\pi^*_{sp} = 1/(\lambda e)$.

If the firm sells the "future" product in advance, it charges its expected valuation in the future: $p^*_{ft} = E(v) = 1/\lambda$. This is identical to the optimal "spot" price; however, the firm can sell to all customers and extract all surplus for profits of $\pi^*_{ft} = \beta/\lambda$, discounted to the current period. It immediately follows that when the discount factor β is high $\left(\beta > \frac{1}{e}\right)$, then $\pi^*_{ft} > \pi^*_{sp}$. With the more general distribution of valuations in our model, the finite-expiration program can outperform the no-expiration program even when purchase frequency and valuation are positively correlated. This is in contrast to (Sun and Zhang, 2019) which finds that a negative correlation is required with a more restrictive distribution.

The reward program with expiring credits can be interpreted as "bundling" two products. As in classic bundled pricing (Adams and Yellen, 1976), the firm can increase profits relative to pricing the products separately by reducing the variance of the distribution of demand. That is, the "future" product makes ex-post customers more similar ex-ante so that the firm does not face as extreme a trade-off between pricing high and losing low-valuation customers versus pricing low and getting more demand. This smoothing of future demand across customers is more important the higher the discount factor for customers (as demonstrated in Proposition 11) since the present value of the effect is higher. That is, the gains from this bundling effect (which derives from customers' future benefit) outweighs the loss from the "down payment" effect only if the discount factor is high enough.

Finally, why does this argument not hold for our baseline example with no expiration? In the finite-expiration case, customers may suffer in the future period when they consume the product: if their valuation ends up being low the value they obtain is lower than the price they paid in advance. That is, they over-consume. However, with no expiration, credits can be used any time in the future, and customers will wait to redeem their credit rather than over-consuming if their realized valuation is low. That is, the finite expiration is necessary to enforce the "bundling" of the products.

6 Conclusion

In this paper, we consider the incentive to introduce a frequency reward program both by a monopolist and under duopoly competition. Most previous studies focus on the effects of a reward program assuming its adoption. The few papers that examine the incentives to introduce a program use somewhat specific assumptions on customer preferences, market participation, or customers' time horizons. We consider a very general setting, nesting common empirical specifications, and find spot pricing dominates under both monopoly and competition. Although we consider a reward program which offers price discounts based on previous cumulative purchases, our analysis also applies to a prepayment program in which customers receive a discount for purchasing above a certain volume but consume some units in the future.[9]

It would be useful to extend the model in order to examine other possible features of reward programs. There are some extensions that could be incorporated in the model relatively easily to determine whether they affect the profitability of a reward program relative to spot pricing. These include behavioral aspects. For example, additional utility from consumption gained from points redemption versus through purchase, and "slippage" from customers failing to redeem earned credits. These would require minimal changes to the model and could be incorporated while maintaining the model's analytical tractability.

It would also be useful to examine how the duration of credit expiration affects reward program profitability. The "bundling" intuition, which explains our finite-expiration results, suggests that a shorter time frame is advantageous, but it would be useful to confirm this. This could potentially be confirmed empirically by comparing the effects of changes in expiration limits in actual programs.

Also for empirical work, our results suggest the need for additional results on the profitability of introducing a reward program – as opposed to examining the effects of an already-established program. There has been more empirical work on the latter than the former. Our results also suggest empirically examining how changing from no- to finite-expiration of credits, or vice versa, affects the profitability of a reward program.

[9] Li et al. (2023) analyze such a prepayment program in a two-period model.

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A Proofs

A.1 Proof of Proposition 2

This continues the proof in the main text. First, solve the steady-state distribution $\{r_s\}$ in the BXGY program:

$$\begin{aligned} r_0 &= q_X r_X + (1-q_0) r_0 && \text{(A1)} \\ r_1 &= q_0 r_0 + (1-q_1) r_1 && \text{(A2)} \\ &\dots && \text{(A3)} \\ r_{X+Y-1} &= q_{X+Y-2} r_{X+Y-2} + (1-q_{X+Y-1}) r_{X+Y-1}, && \text{(A4)} \end{aligned}$$

where

$$q_s = G(p \cdot I(s < X) - \beta \Delta w(s)). \tag{A5}$$

This is a system of linear equations, which has one unique solution:

$$r_s = \frac{1}{q_s} \cdot \left(\sum_{s=0}^{X+Y-1} \frac{1}{q_s} \right)^{-1}. \tag{A6}$$

Consequently, the demand function:

$$D_{RX}(p) = \sum_{s=0}^{X-1} r_s \cdot q_s = X \cdot \left(\sum_{s=0}^{X+Y-1} \frac{1}{q_s} \right)^{-1}. \tag{A7}$$

Following similar logic to that in Proposition 1, since $\sum_s \Delta w(s) = 0$ and $H(\cdot)$ is convex, we can apply Jensen's inequality to get:

$$\frac{1}{X+Y} \left(\sum_{s=0}^{X+Y-1} H(p \cdot I(s < X) - \beta \Delta w(s)) \right) \geq H\left(\frac{X \cdot p}{X+Y} \right). \tag{A8}$$

Plugging back into Equations (A5) and (A7), spot pricing demand exceeds reward program demand.

A.2 Proof of Proposition 3

The customers' payoffs are:

$$\begin{aligned} u_1(v,s) &= v - p_s + \beta \cdot w(1-s) && \text{(A9)} \\ u_0(v,s) &= \beta \cdot w(s), && \text{(A10)} \end{aligned}$$

with purchase probabilities:

$$\begin{aligned} q_0 &= \Pr(u_1(v,0) > u_0(v,0)) = G(p_0 - \beta \Delta w) && \text{(A11)} \\ q_1 &= \Pr(u_1(v,1) > u_0(v,1)) = G(p_1 + \beta \Delta w), && \text{(A12)} \end{aligned}$$

and $\Delta w = w(1) - w(0)$. Let $\pi_R(p_0, p_1)$ be the per-period demand under this program, we have the following results.

The steady-state distribution under credit-specific pricing takes the same form as in (9), which means

$q_0 r_0 = q_1 r_1 = \left(\frac{1}{q_0} + \frac{1}{q_1}\right)^{-1}$ in the case with $X = 1$ and $Y = 1$, so:

$$\pi_R(p_0, p_1) = p_0 q_0 r_0 + p_1 r_1 q_1 = 2 \cdot \bar{p} \cdot \left(\frac{1}{q_0} + \frac{1}{q_1}\right)^{-1}, \tag{A13}$$

where $\bar{p} = (p_0 + p_1)/2$. Applying a procedure analogous to that in Proposition 1 shows the result.

A.3 Proof of Proposition 5

For what follows, we impose a mild regularity condition that guarantees the existence of the market (customers with the highest valuation buy):

Assumption 2. *The support of v is unbounded:*

$$\sup v = +\infty.$$

The value function is determined as (write $t_1 = 1 - t$)

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = E \max \begin{pmatrix} v - (p+t) + \beta w_1, & v - (p_1 + t_1) + \beta w_0, & \beta w_0 \\ v - t + \beta w_0, & v - (p_1 + t_1) + \beta w_1, & \beta w_1 \end{pmatrix}, \tag{A14}$$

which implies:

$$(1-\beta) w_0 = E \max(v - (p+t) + \beta \Delta w, v - (p_1 + t_1), 0) \tag{A15}$$

$$(1-\beta) w_1 = E \max(v - t - \beta \Delta w, v - (p_1 + t_1), 0). \tag{A16}$$

We prove both claims by contradiction.

(a) Suppose $w_1 < w_0$, then $\Delta w < 0$. Combining this with the regularity assumption in 2, the right-hand side of Equation (A15) is smaller than the right-hand side of Equation (A16). This implies $w_0 \leq w_1$, which is a contradiction.

(b) Suppose $\beta \Delta w > \frac{p}{2}$, then $v - (p+t) + \beta \Delta w > v - t - \beta \Delta w$. This implies $w_0 \geq w_1$, which is a contradiction.

A.4 Proof of Proposition 7

Re-write the one-step contraction mapping T for $w = (w_{00}, w_{10}, w_{01}, w_{11})^T$ and let $q = p_1$ and $s = 1 - t$,

$$T(w) = T \begin{pmatrix} w_{00} \\ w_{10} \\ w_{01} \\ w_{11} \end{pmatrix} = E \max \begin{pmatrix} v - (p+t) + \beta w_{10}, & v - (q+s) + \beta w_{01}, & \beta w_{00} \\ v - t + \beta w_{00}, & v - (q+s) + \beta w_{11}, & \beta w_{10} \\ v - (p+t) + \beta w_{11}, & v - s + \beta w_{00}, & \beta w_{01} \\ v - t + \beta w_{01}, & v - s + \beta w_{10}, & \beta w_{11} \end{pmatrix}. \tag{A17}$$

In order to prove the result, we first establish some properties of the value function.

A.4.1 Math preliminaries

Proposition 12. *If w satisfies the following properties:*

$$0 \leq w_{10} - w_{00} \leq p \tag{A18}$$

$$0 \leq w_{11} - w_{01} \leq p \tag{A19}$$

and:

$$w' = T(w), \tag{A20}$$

then w' also satisfies the above property. Similar arguments apply for the two properties

$$0 \leq w_{01} - w_{00} \leq q \tag{A21}$$
$$0 \leq w_{11} - w_{10} \leq q. \tag{A22}$$

As the value function is the limit of the contraction mapping, and the starting vector $w^{(0)} = 0$ satisfies all four properties, so is the limit.

Proof. We only prove the first part relating to $(w_{10} - w_{00})$ and $(w_{11} - w_{01})$. The second part is similar.

(a) Since $0 \leq w_{10} - w_{00} \leq p$, and $\beta \in (0, 1)$, we have $\beta(w_{10} - w_{00}) \leq p$. Thus $\beta w_{10} \leq \beta w_{00} + p$. Similarly, for $\beta w_{11} \leq \beta w_{01} + p$.

(b) Applying Equation (A19) and step (a) above, $w'_{00} \leq w'_{10}$:

$$w'_{00} = E \max(v - (p + t) + \beta w_{10}, v - (q + s) + \beta w_{01}, \beta w_{00}) \tag{A23}$$
$$\leq E \max(v - t + \beta w_{00}, v - (q + s) + \beta w_{11}, \beta w_{10}) = w'_{10}. \tag{A24}$$

(c) Applying Equation (A18) and step (a) above, $w'_{10} - w'_{00} \leq p$:

$$w'_{10} - p = E \max(v - (p + t) + \beta w_{00}, v - (q + s) + \beta w_{11} - p, \beta w_{10} - p) \tag{A25}$$
$$\leq E \max(v - (p + t) + \beta w_{10}, v - (q + s) + \beta w_{01}, \beta w_{00}) = w'_{00}. \tag{A26}$$

(d) Applying Equation (A19) and step (a) above, $w'_{01} \leq w'_{11}$:

$$w'_{01} = E \max(v - (p + t) + \beta w_{11}, v - s + \beta w_{00}, \beta w_{01}) \tag{A27}$$
$$\leq E \max(v - t + \beta w_{01}, v - s + \beta w_{10}, \beta w_{11}) = w'_{11}. \tag{A28}$$

(e) Applying Equation (A18) and step (a) above, $w'_{11} - w'_{01} \leq p$:

$$w'_{11} - p = E \max(v - (p + t) + \beta w_{01}, v_t - s + \beta w_{10} - p, \beta w_{11} - p) \tag{A29}$$
$$\leq E \max(v - (p + t) + \beta w_{11}, v_t - s + \beta w_{00}, \beta w_{01}) = w'_{01}. \tag{A30}$$

□

Proposition 13. *The value function determined by the fixed point of $T(\cdot)$ satisfies*

$$0 \leq w_{11} - w_{00} \leq \max(q + s - t, p + t - s). \tag{A31}$$

Proof. It takes several steps.

(a) $w_{11} - w_{00} = w_{11} - w_{01} + w_{01} - w_{00} \geq 0$. This follows from Equations (A19) and (A21).

(b) To prove that $w_{11} - w_{00} \leq \max(q + s - t, p + t - s)$, we show that for any w that satisfies (A31), $w' = T(w)$, That is, $w'_{11} - \max(q + s - t, p + t - s) \leq w'_{00}$:

$$v - t + \beta w_{01} - \max(q + s - t, p + t - s) \leq v - (q + s) + \beta w_{01} \tag{A32}$$
$$v - s + \beta w_{10} - \max(q + s - t, p + t - s) \leq v - (p + t) + \beta w_{10} \tag{A33}$$
$$\beta w_{11} - \max(q + s - t, p + t - s) \leq \beta w_{00}. \tag{A34}$$

where the first inequality follows from the fact that $-\max(q+s-t, p+t-s) \leq -(q+s-t)$, the second inequality follows from the fact that $-\max(q+s-t, p+t-s) \leq -(p+t-s)$, the last inequality follows from Equation (A31). Taking the $E\max(\cdot)$ of the three terms on the left and on the right, respectively, and apply the definition of contraction mapping in (A17):

$$w'_{11} - \max(q+s-t, p+t-s) \leq w'_{00}. \tag{A35}$$

□

We also prove one Lemma that is useful in the follow-up proofs.

Lemma 1. *For any two numbers $\{a, b\}$, if $E\max(v-b, 0) \leq E\max(v-a, 0)$, then*

$$E\max(v-a, 0) - E\max(v-b, 0) \leq b-a. \tag{A36}$$

Proof. We first show $b \geq a$. Suppose not, $a > b$, then $\max(v-b, 0) \geq \max(v-a, 0)$, and with strict inequality for $v > a$. By Assumption 2, that is in the support of v, thus $E\max(v-b, 0) > E\max(v-a, 0)$, contracts with the condition.

Thus, $E\max(v-a, 0) - (b-a) = E\max(v-b, a-b) \leq E\max(v-b, 0)$. □

A.4.2 Proof of $A_{10} \leq A_{00}$

Proof. We prove by contradiction. Suppose $A_{10} > A_{00}$, then $A_{10} = 1$, $A_{00} = 0$, which implies:

$$\beta(w_{11} - w_{00}) \leq q+s-t \tag{A37}$$

$$\beta(w_{10} - w_{01}) \geq (p+t) - (q+s). \tag{A38}$$

(a) $q+s-t < p+t-s$. Suppose not, then by (A31), $\beta w_{11} - \beta w_{00} \leq q+s-t$, this implies $A_{10} = 0$.

(b) Using the previous step, $A_{01} = 1$, as (A31), $w_{11} - w_{00} \leq \max(q+s-t, p+t-s) = p+t-s$.

(c) $A_{11} = 1$, as by (A38), $\beta(w_{10} - w_{01}) \geq (p+t) - (q+s) \geq s-t$ where the last inequality follows from step (a) above. Therefore, $\beta w_{10} - s \geq \beta w_{01} - t$.

(d) $w_{10} \leq w_{01}$, since:

$$w_{10} = E\max(v - (q+s) + \beta w_{11}, \beta w_{10}) \tag{A39}$$

$$w_{01} = E\max(v - s + \beta w_{00}, \beta w_{01}). \tag{A40}$$

The first line follows from the assumption that $A_{10} = 1$ and the second line from step (b) above ($A_{01} = 1$). Furthermore, from step (b) above, we have:

$$\beta(w_{11} - w_{00}) \leq p+t-s. \tag{A41}$$

Manipulating Equation (A39), we have:

$$w_{10} - \beta(w_{10} - w_{01}) = E\max(v - (q+s) + \beta w_{11} - \beta(w_{10} - w_{01}), \beta w_{01}). \tag{A42}$$

Plugging in Equation (A38):

$$w_{10} - \beta(w_{10} - w_{01}) \leq E\max(v + \beta w_{11} - (p+t), \beta w_{01}), \tag{A43}$$

and plugging in Equation (A41) and (A40):

$$E\max(v + \beta w_{11} - (p+t), \beta w_{01}) \leq E\max(v - s + \beta w_{00}, \beta w_{01}) = w_{01}. \tag{A44}$$

Therefore, combining Equation (A43) and Equation (A44):

$$(1-\beta)(w_{10}-w_{01}) \leq 0, \tag{A45}$$

which implies

$$w_{10} \leq w_{01}. \tag{A46}$$

(e) Thus, we have:

$$\beta w_{00} \leq \beta w_{10} \leq \beta w_{01} \leq \beta w_{11}., \tag{A47}$$

where the first inequality follows from Equation (A18), the second inequality from Equation (A46), and the third inequality from Equation (A19). Define:

$$x = \beta w_{10} - \beta w_{00} \tag{A48}$$

$$y = \beta w_{01} - \beta w_{10} \tag{A49}$$

$$z = \beta w_{11} - \beta w_{01}. \tag{A50}$$

From $A_{00} = 0$ and $A_{10} = 1$, plug in (A37) and (A38)

$$y \leq (q+s) - (p+t) \tag{A51}$$

$$x + y + z \geq q + s - t. \tag{A52}$$

(f) The value functions are (since $A_{00} = 0$, $A_{10} = 1$, $A_{01} = 1$ (see step (b) above), and $A_{11} = 1$ (see step (c) above), respectively):

$$(1-\beta)w_{00} = E\max(v-(p+t)+\beta(w_{10}-w_{00}),0) = E\max(v-(p+t)+x,0) \tag{A53}$$

$$(1-\beta)w_{10} = E\max(v-(q+s)+\beta(w_{11}-w_{10}),0) = E\max(v-(q+s)+y+z,0) \tag{A54}$$

$$(1-\beta)w_{01} = E\max(v-s-\beta(w_{01}-w_{00}),0) = E\max(v-s-x-y,0) \tag{A55}$$

$$(1-\beta)w_{11} = E\max(v-s-\beta(w_{11}-w_{10}),0) = E\max(v-s-y-z,0), \tag{A56}$$

Using the above and applying the result from Lemma 1 (the precondition is satisfied as $w_{00} \leq w_{10}$ and $w_{01} \leq w_{11}$):

$$(1-\beta)(w_{10}-w_{00}) \leq (p+t-x)-(q+s-y-z) \leq z-x \tag{A57}$$

$$(1-\beta)(w_{11}-w_{01}) \leq (s+x+y)-(s+y+z) = x-z, \tag{A58}$$

where the last inequality in the first line follows from Equation (A51). This means, by Equation (A47), that $x = z = 0$. By Equations (A52) and (A51) in turn, that $q+s-t \leq x+y+z = y \leq (q+s)-(p+t)$. This is a contradiction.

□

A.4.3 Proof of $A_{10} \leq A_{11}$

Proof. Again, we prove by contradiction. Suppose $A_{10} > A_{11}$, then $A_{10} = 1$, $A_{11} = 0$, and:

$$\beta(w_{01}-w_{10}) \geq t-s. \tag{A59}$$

(a) $q+s-t < p+t-s$. Suppose not, then by (A31), $\beta w_{11} - \beta w_{00} \leq q+s-t$, this implies $A_{10} = 0$.

(b) $A_{01} = 1$, as by (A31), $w_{11} - w_{00} \leq \max(q+s-t, p+t-s) = p+t-s$.

(c) $A_{00} = 1$, as by (A59), $\beta(w_{01}-w_{10}) \geq t-s \geq (q+s)-(p+t)$, where the last inequality follows from step (a) above.

(d) $w_{01} \leq w_{10}$, since:

$$w_{10} = E\max(v-(q+s)+\beta w_{11}, \beta w_{10}) \tag{A60}$$
$$w_{01} = E\max(v-s+\beta w_{00}, \beta w_{01}). \tag{A61}$$

The first line follows from the assumption that $A_{10}=1$ and the second line from the assumption that $A_{01}=1$. Furthermore, as $A_{10}=1$, we have:

$$\beta(w_{11}-w_{00}) \geq (q+s)-t. \tag{A62}$$

Using Equations (A59), (A62) and (A60) sequentially:

$$w_{01}-\beta(w_{01}-w_{10}) \tag{A63}$$
$$\leq E\max(v-t+\beta w_{00}, \beta w_{10}) \tag{A64}$$
$$\leq E\max(v-(q+s)+\beta w_{11}, \beta w_{10}) = w_{10}, \tag{A65}$$

where the last inequality follows from Equation (A62) and the last equality follows from $A_{10}=1$. Therefore:

$$(1-\beta)(w_{01}-w_{10}) \leq 0 \tag{A66}$$

which implies:

$$w_{01} \leq w_{10}. \tag{A67}$$

(e) Thus, we have

$$\beta w_{00} \leq \beta w_{01} \leq \beta w_{10} \leq \beta w_{11}, \tag{A68}$$

where the first inequality follows from Equation (A18), the second inequality from the equation directly above, and the third inequality from Equation (A22). Define:

$$x = \beta w_{01} - \beta w_{00} \tag{A69}$$
$$y = \beta w_{10} - \beta w_{01} \tag{A70}$$
$$z = \beta w_{11} - \beta w_{10}. \tag{A71}$$

From (A62), we have

$$x+y+z \geq q+s-t. \tag{A72}$$

(f) The value functions are (since $A_{00}=1$ and $A_{11}=0$ respectively):

$$(1-\beta)w_{00} = E\max(v-(q+s)+\beta(w_{01}-w_{00}),0) = E\max(v-(q+s)+x,0) \tag{A73}$$
$$(1-\beta)w_{11} = E\max(v-t-\beta(w_{11}-w_{01}),0) = E\max(v-t-y-z,0), \tag{A74}$$

Applying the result from Lemma 1:

$$(1-\beta)(w_{11}-w_{00}) \leq (q+s-x)-(t+y+z) = (q+s-t)-\beta(w_{11}-w_{00}), \tag{A75}$$

which means:

$$w_{11}-w_{00} \leq q+s-t. \tag{A76}$$

and given $w_{00} \leq w_{11}$:

$$\beta\,(w_{11}-w_{00}) \leq q+s-t. \tag{A77}$$

This contradicts (A62).

□

A.4.4 Other parts of the proposition

Showing that $A_{00} \leq A_{01}$ is identical to the proof of $A_{10} \leq A_{00}$ with the two firms interchanged. Similarly, proving $A_{11} \leq A_{01}$ is identical to proving $A_{10} \leq A_{11}$ with the two firms interchanged.

A.5 Proof of Proposition 8

Proof. As $w = (w_{10}, w_{10}, w_{01}, w_{11})^T$ is the fixed point of the following contraction mapping:

$$T(w) = E\max\begin{pmatrix} v-(p+t)+\beta w_{10}, & v-1-q+t+\beta w_{01}, & \beta w_{00} \\ v-t+\beta w_{00}, & v-1-q+t+\beta w_{11}, & \beta w_{10} \\ v-(p+t)+\beta w_{11}, & v-1+t+\beta w_{00}, & \beta w_{01} \\ v-t+\beta w_{01}, & v-1+t+\beta w_{10}, & \beta w_{11} \end{pmatrix}, \tag{A78}$$

and:

$$A_{00} = I(2t + \beta w_{01} - \beta w_{10} \leq q - p + 1). \tag{A79}$$

Therefore, it suffices to show:

$$\psi(t) = 2t + \beta w_{01}(t) - \beta w_{10}(t), \tag{A80}$$

increases in t.

As all elements in the contraction mapping are continuous and $E\max(\cdot)$ preserves continuity, the value function (fixed point) is continuous in t. Therefore, so is $\psi(t)$. Since $A_{10} \leq A_{00} \leq A_{01}$ by Proposition 7, we consider the following four cases and show that $\psi(t)$ is increasing in all four:[10]

1. $A_{10} = A_{00} = A_{01} = 0$
2. $A_{10} = A_{00} = 0,\ A_{01} = 1$
3. $A_{10} = A_{00} = A_{01} = 1$
4. $A_{10} = 0,\ A_{00} = A_{01} = 1$

A.5.1 Case 1: $A_{10} = A_{00} = A_{01} = 0$

Since $A_{11} \leq A_{01}$ by Proposition 7, $A_{11} = 0$. In this case, the value function is also the fixed point of the following mapping:

$$T(w) = E\max\begin{pmatrix} v-(p+t)+\beta w_{10}, & \beta w_{00} \\ v-t+\beta w_{00}, & \beta w_{10} \\ v-(p+t)+\beta w_{11}, & \beta w_{01} \\ v-t+\beta w_{01}, & \beta w_{11} \end{pmatrix}, \tag{A81}$$

so that $w_{00} = w_{01} = w_0$, $w_{10} = w_{11} = w_1$, and $\psi(t) = 2t + \beta w_0 - \beta w_1$. The contraction mapping collapses to:

$$w_0' = E\max(v - p - t + \beta w_1, \beta w_0) \tag{A82}$$
$$w_1' = E\max(v - t + \beta w_0, \beta w_1). \tag{A83}$$

Define $\Delta w = w_1 - w_0$. We show that if $t + \Delta w$ and $t - \Delta w$ are increasing in t, then $t + \Delta w'$ and $t - \Delta w'$ are also increasing in t. This then implies that this holds in the limit as well and $\psi(t) = (2-\beta)t + \beta(t + w_0 - w_1)$ increases in t.

[10]Let the domain of $\psi(t)$, the unit interval, be the union of mutually-disjoint compact sets (i.e., $[0,1] = I_1 \cup I_2 ... \cup I_K$). To show that $\psi(t)$ is increasing in t, it suffices to show it increases in t in each I_k.

These increasing properties hold since:

$$t+\Delta w' = t + E\max(v-t+\beta w_0, \beta w_1) - E\max(v-p-t+\beta w_1, \beta w_0) \quad \text{(A84)}$$
$$= E\max(v, t+\beta\Delta w) - E\max(v-p-t+\beta\Delta w, 0) \quad \text{(A85)}$$
$$t-\Delta w' = t + E\max(v-p-t+\beta w_1, \beta w_0) - E\max(v-t+\beta w_0, \beta w_1) \quad \text{(A86)}$$
$$= E\max(v-p, t-\beta\Delta w) - E\max(v-t-\beta\Delta w, 0), \quad \text{(A87)}$$

both increase in t.

A.5.2 Case 2: $A_{10} = A_{00} = 0$, $A_{01} = 1$

In this case, A_{11} can be disregarded since $A_{10} = 0$ and $A_{01} = 1$. Thus, the value function is also the fixed point of the following mapping:

$$w'_{00} = E\max(v-p-t+\beta w_{10}, \beta w_{00}) \quad \text{(A88)}$$
$$w'_{10} = E\max(v-t+\beta w_{00}, \beta w_{10}) \quad \text{(A89)}$$
$$w'_{01} = E\max(v-1+t+\beta w_{00}, \beta w_{01}). \quad \text{(A90)}$$

That $t \pm (w_{10} - w_{00})$ increases in t follows directly from above. Suppose that $t + (w_{01} - w_{10})$ increases in t. We can show that the contraction mapping is also increasing in t:

$$t + w'_{01} - w'_{10} \quad \text{(A91)}$$
$$= t + E\max(v-1+t+\beta w_{00}, \beta w_{01}) - E\max(v-t+\beta w_{00}, \beta w_{10}) \quad \text{(A92)}$$
$$= E\max(v-1+2t+\beta(w_{00}-w_{10}), t+\beta(w_{01}-w_{10})) - E\max(v-t-\beta(w_{10}-w_{00}), 0) \quad \text{(A93)}$$

which increases in t. Therefore, in the limit so does $\psi(t)$.

A.5.3 Case 3: $A_{10} = A_{00} = A_{01} = 1$

Since $A_{11} \geq A_{10}$ by Proposition 7, $A_{11} = 1$. Therefore, $w_{00} = w_{10} = w_0$, $w_{01} = w_{11} = w_1$. Define $u = v - 1$. $(w_0, w_1)^T$ is the fixed point of the following contraction mapping:

$$w'_0 = E\max(u-q+t+\beta w_1, \beta w_0) \quad \text{(A94)}$$
$$w'_1 = E\max(u+t+\beta w_0, \beta w_1). \quad \text{(A95)}$$

Define $\Delta w = w_1 - w_0$. Suppose $t \pm \Delta w$ increases in t, then:

$$t + w'_0 - w'_1 = E\max(u-q+t+\beta\Delta w, 0) - E\max(u, -t+\beta\Delta w) \quad \text{(A96)}$$
$$t + w'_1 - w'_0 = E\max(u+t-\beta\Delta w, 0) - E\max(u-q, -t-\beta\Delta w), \quad \text{(A97)}$$

both increase in t.

A.5.4 Case 4: $A_{10} = 0$, $A_{00} = A_{01} = 1$

In this case, A_{11} can be disregarded since $A_{10} = 0$ and $A_{01} = 1$. Then $(w_{00}, w_{01}, w_{10})^T$ is the limit of the following contraction mapping

$$w'_{00} = E\max(u-q+t+\beta w_{01}, \beta w_{00}) \quad \text{(A98)}$$
$$w'_{10} = E\max(v-t+\beta w_{00}, \beta w_{10}) \quad \text{(A99)}$$
$$w'_{01} = E\max(u+t+\beta w_{00}, \beta w_{01}). \quad \text{(A100)}$$

That $t \pm (w_{00} - w_{01})$ increases in t follows directly from the above. Suppose $t + w_{01} - w_{10}$ increases in t. Then we can show the contraction mapping is also increasing in t:

$$\begin{aligned} & t + w'_{01} - w'_{10} && \text{(A101)} \\ =& t + E\max(u + t + \beta w_{00}, \beta w_{01}) - E\max(v - t + \beta w_{00}, \beta w_{10}) && \text{(A102)} \\ =& E\max(u + t + \beta(w_{00} - w_{01}), 0) - E\max(v - 2t + \beta(w_{00} - w_{01}), -t - \beta(w_{01} - w_{10})). && \text{(A103)} \end{aligned}$$

Therefore, $t + w'_{01} - w'_{10}$ also increases in t.

□

A.6 Proof of Proposition 9

Proof. The proof is a simplified version of Propositions 7 and 8. Define $q = p_1$, $s = 1 - t$, and (u_0, u_1) as the limit of the following contraction mapping:

$$u'_0 = E\max(v - (p + t) + \beta u_0, v - (q + s) + \beta u_1, \beta u_0) \quad \text{(A104)}$$
$$u'_1 = E\max(v - (p + t) + \beta u_1, v - s + \beta u_0, \beta u_1). \quad \text{(A105)}$$

A.6.1 $u_0 \leq u_1$ and $\beta(u_1 - u_0) \leq \frac{q}{2}$

In the limit:

$$u_0 = E\max(v - (p + t) + \beta u_0, v - (q + s) + \beta u_1, \beta u_0) \quad \text{(A106)}$$
$$u_1 = E\max(v - (p + t) + \beta u_1, v - s + \beta u_0, \beta u_1). \quad \text{(A107)}$$

Therefore:

$$(1 - \beta)u_0 = E\max(v - (p + t), v - (q + s) + \beta\Delta u, 0) \quad \text{(A108)}$$
$$(1 - \beta)u_1 = E\max(v - (p + t), v - s - \beta\Delta u, 0), \quad \text{(A109)}$$

where $\Delta u = u_1 - u_0$.

(a) Suppose $u_1 < u_0$, then $\Delta u < 0$ and $(1 - \beta)\, u_0 \leq (1 - \beta)\, u_1$, which is a contradiction.

(b) Suppose $\beta\Delta u > \frac{q}{2}$, then $u_0 \geq u_1$. However, since $\beta\Delta u > \frac{q}{2} > 0$, $u_1 > u_0$. This is a contradiction.

A.6.2 $A_0^{SR} \leq A_1^{SR}$

For simplicity, we suppress the superscript and write A_0 and A_1 as:

$$A_0 = I(-(q + s) + \beta u_1 \geq -(p + t) + \beta u_0) = I(q - \beta\Delta u \leq p + t - s) \quad \text{(A110)}$$
$$A_1 = I(-s + \beta u_0 \geq -(p + t) + \beta u_1) = I(\beta\Delta u \leq p + t - s). \quad \text{(A111)}$$

Since $\beta\Delta u \leq \frac{q}{2}$,$\beta\Delta u \leq q - \beta\Delta u$, so $A_1 \geq A_0$.

A.6.3 A_0 increases in t

It suffices to show:

$$\psi(t) = 2t + \beta(u_1 - u_0), \quad \text{(A112)}$$

increases in t following the same argument as below Equation A83.

(a) $A_0 = A_1 = 0$. In this case, $u_0 = u_1$, and $\psi(t)$ increases in t.

(b) $A_0 = A_1 = 1$. In this case, the value function is equivalent to the case in Proposition 8 with $A_{10} = A_{00} = A_{01} = 1$.

(c) $A_0 = 0$, and $A_1 = 1$. So

$$u_0' = E\max(v - p - t + \beta u_0, \beta u_0) \tag{A113}$$
$$u_1' = E\max(v - 1 + t + \beta u_0, \beta u_1). \tag{A114}$$

Suppose $t + \Delta u$ increases in t, then:

$$t + \Delta u' = t + E\max(v - 1 + t + \beta u_0, \beta u_1) - E\max(v - p - t + \beta u_0, \beta u_0) \tag{A115}$$
$$= E\max(v - 1 + 2t, t + \beta\Delta u) - E\max(v - p - t, 0) \tag{A116}$$

which also increases in t.

□

A.7 Proof of Proposition 10

A.7.1 Preliminaries

We first prove a property of a value function when customers are restricted to choose between firm 0 and no purchase. This property helps in the proof of the proposition, as it turns out to be the value function for customers at cutoff-type $\bar{t}$.

Lemma 2. *If the value function is determined by*

$$w_{00} = E\max(v - (p + t) + \beta w_{10}, \beta w_{00}) \tag{A117}$$
$$w_{10} = E\max(v - \bar{t} + \beta w_{00}, \beta w_{10}), \tag{A118}$$

then

$$\beta(w_{10} - w_{00}) \leq \frac{p}{2}. \tag{A119}$$

Proof. The proof is largely similar to that of Proposition 5. The two equations can be represented as

$$(1 - \beta) \cdot w_{00} = E\max(v - (p + t) + \beta\Delta w_0, 0) \tag{A120}$$
$$(1 - \beta) \cdot w_{10} = E\max(v - \bar{t} - \beta\Delta w_0, 0), \tag{A121}$$

where $\Delta w_0 = w_{10} - w_{00}$. The following statements are proven by contradiction:

1. $\Delta w_0 \geq 0$. If not, the right-hand-side of the first equation is larger than that of the second one, implying $(1 - \beta)w_{00} > (1 - \beta)w_{10}$, which is a contradiction.

2. $\Delta w_0 \leq \frac{p}{2}$. If not, the right-hand side of the first equation is larger than that of the second, which is a contradiction.

□

A.7.2 Proposition proof

Proof. We first show that $C_{00}^{RR}(p, t) \leq C_0^{SR}(\frac{p}{2}, t)$. As customers' choice of firm 1 under either market structure ($A_{00}(\cdot)$ and $A_0(\cdot)$ given by Equations (A79) and (A110) respectively), are both increasing in t, it suffices to show that the threshold type ($\bar{t}$) defined in (58) (the type that is indifferent between the two firms if firm 0 offers a reward program with price p) purchases from firm 0 if firm 0 charges a spot price of $\frac{p}{2}$. That is, $A_0(\frac{p}{2}, \bar{t}) = 0$.

Rewrite the definition of $\bar{t}\ (w)$ when firm 0 offers a reward program at price p:

$$-(p+\bar{t})+\beta w_{10} = -(p_1+1-\bar{t})+\beta w_{01}. \tag{A122}$$

Therefore:

$$w_{00} = E\max(v-(p_1+1-\bar{t})+\beta w_{01}, \beta w_{00}) \tag{A123}$$
$$w_{01} = E\max(v-(1-\bar{t})+\beta w_{00}, \beta w_{01}). \tag{A124}$$

Suppose this customer chooses firm 1 under market structure SR $\left(A_0^{SR}(\frac{p}{2},\bar{t})=1\right)$:

$$-(p_1+1-\bar{t})+\beta u_1 > -(\frac{p}{2}+t)+\beta u_0, \tag{A125}$$

where:

$$u_0 = E\max(v-(p_1+1-\bar{t})+\beta u_1, \beta u_0) \tag{A126}$$
$$u_1 = E\max(v-(1-\bar{t})+\beta u_0, \beta u_1). \tag{A127}$$

Comparing the two sets of value functions, we have:

$$w_{00} = u_0 \tag{A128}$$
$$w_{01} = u_1. \tag{A129}$$

Substituting into Equation (A125):

$$-(p_1+1-\bar{t})+\beta w_{01} > -(\frac{p}{2}+\bar{t})+\beta w_{00}. \tag{A130}$$

On the other hand, as type $\bar{t}$ customer is indifferent in choosing firm 0 and firm 1 at state (00), $A_{10}=0$, and:

$$w_{00} \quad = \quad E\max(v-(p+\bar{t})+\beta w_{10}, \beta w_{00}) \tag{A131}$$
$$w_{10} \quad = \quad E\max(v-\bar{t}+\beta w_{00}, \beta w_{10}). \tag{A132}$$

These can be transformed to:

$$(1-\beta)w_{00} \quad = \quad E\max(v-(p+\bar{t})+\beta\Delta w, 0) \tag{A133}$$
$$(1-\beta)w_{10} \quad = \quad E\max(v-\bar{t}-\beta\Delta w, 0), \tag{A134}$$

where $\Delta w = w_{10}-w_{00}$. Suppose $\beta\Delta w > \frac{p}{2}$. Then $v-(p+\bar{t})+\beta\Delta w > v-\bar{t}-\beta\Delta w$. This implies $w_{00} > w_{10}$, which is a contradiction.

Combining this with Equation A130:

$$-(p_1+1-\bar{t})+\beta w_{01} > -(\frac{p}{2}+\bar{t})+\beta w_{00} \geq -(p+\bar{t})+\beta w_{10}, \tag{A135}$$

This contradicts Equation (A122). Therefore, we have:

$$C_{00}^{RR}(p,t) \leq C_0^{SR}(\frac{p}{2},t). \tag{A136}$$

That $D_{RR}(p,t) \leq \frac{1}{2}D_{SR}(\frac{p}{2},t)$ follows directly from Proposition 1 of the monopoly case since $\frac{q_1(t)q_0(t)}{q_1(t)+q_0(t)}$, defined in Equation (59), is less than $\frac{1}{2}G(\frac{p}{2}+t)$, defined in Equation (63).

□

A.8 Proof of Proposition 11

Proof. We first show some mathematical preliminaries relating to the exponential distribution before deriving the optimal spot pricing p_S^*. We then show that the firm can earn higher profits under the reward program than under spot pricing $\left(\pi_R^F(2p_S^*) > \pi_S^*\right)$ by charging $2p_S^*$.

A.8.1 Preliminaries

If $v \sim Exp(\lambda)$, then:

$$E\max(v-b,0) = \frac{1}{\lambda}e^{-\lambda b}. \tag{A137}$$

If $x \sim Exp(\lambda)$ then:

$$\int_b^{\infty} x\cdot\lambda\exp(-\lambda x)\,dx = \int_0^{\infty}(y+b)\cdot\lambda\cdot\exp(-\lambda(y+b))\,dy \tag{A138}$$

$$= e^{-\lambda b}\cdot\left(\int_0^{\infty} y\cdot\lambda\exp(-\lambda y)\,dy + \int_0^{\infty} b\cdot\lambda\exp(-\lambda y)\,dy\right) \tag{A139}$$

$$= e^{-\lambda b}\cdot\left(\frac{1}{\lambda}+b\right), \tag{A140}$$

so:

$$E\max(x,b) = \int_0^{b} b\cdot\lambda\exp(-\lambda x)\,dx + \int_b^{\infty} x\cdot\lambda\exp(-\lambda x)\,dx \tag{A141}$$

$$= b\cdot\left(1-e^{-\lambda b}\right) + e^{-\lambda b}\left(\frac{1}{\lambda}+b\right) = b + \frac{1}{\lambda}e^{-\lambda b}, \tag{A142}$$

and:

$$E\max(x-b,0) = E\max(x,b) - b = \frac{1}{\lambda}\exp(-\lambda b). \tag{A143}$$

A.8.2 Optimal spot pricing

For a firm using spot pricing:

$$\max_p p\cdot G(p) = \max\left[p\cdot\exp(-\lambda p)\right]. \tag{A144}$$

The first-order condition implies:

$$p_S^* = \frac{1}{\lambda}, \tag{A145}$$

and profits are:

$$\pi_S^* = \frac{1}{\lambda e}. \tag{A146}$$

A.8.3 Value function under the reward program

The value function under the reward program is determined by:

$$w_0 = E\max(v-p+\beta w_1, \beta w_0) = E\max(v-p+\beta\Delta w, 0) + \beta w_0 \tag{A147}$$

$$w_1 = E(v) + \beta w_0 = \frac{1}{\lambda} + \beta w_0. \tag{A148}$$

Taking the difference between the two:

$$\Delta w = \frac{1}{\lambda} - E \max\left(v - p + \beta \Delta w, 0\right) = \frac{1}{\lambda} - \frac{1}{\lambda} \exp\left(-\lambda\left(p - \beta \Delta w\right)\right), \tag{A149}$$

which gives:

$$1 - \lambda \Delta w = \exp\left(-\lambda\left(p - \beta \Delta w\right)\right). \tag{A150}$$

Finally, we show that Δw, the solution to the above equation, decreases in β. Suppose not, there exists $\beta_1 < \beta_2$ such that $\Delta w_1 < \Delta w_2$, where:

$$1 - \lambda \Delta w_1 = \exp\left(-\lambda(p - \beta_1 \Delta w_1)\right) \tag{A151}$$

$$1 - \lambda \Delta w_2 = \exp\left(-\lambda(p - \beta_2 \Delta w_2)\right). \tag{A152}$$

So far we have $\Delta w_1 < \Delta w_2$. From the left-hand sides of Equations (A151) and (A152), $\exp\left(-\lambda(p - \beta_1 \Delta w_1)\right) > \exp\left(-\lambda(p - \beta_2 \Delta w_2)\right)$. This means $\beta_1 \Delta w_1 > \beta_2 \Delta w_2$, which is a contradiction.

A.8.4 Profit under reward program

From Equation (81) in the main text, profit under the reward program is:

$$\pi_R^F(p) = p \cdot \frac{G\left(p - \beta \Delta w\right)}{1 + G\left(p - \beta \Delta w\right)}. \tag{A153}$$

Since $G\left(p - \beta \Delta w\right) = \exp\left(-\lambda(p - \beta \Delta w)\right)$ and substituting from Equation (A150), we have:

$$\pi_R^F\left(p, \Delta w(p, \beta)\right) = p \cdot \frac{1 - \lambda \Delta w}{2 - \lambda \Delta w}, \tag{A154}$$

which decreases in Δw, and thus increases in β.

A.8.5 Comparing two profits

To prove the proposition, it suffices to show that for one price $p_0 = \frac{2}{\lambda}$ (which is $2 \cdot p_S^*$):

$$\pi_R^F(p_0, \Delta w(p_0, 1)) > \frac{1}{\lambda e} = \pi_S^*. \tag{A155}$$

Equivalently, for $u = \lambda \cdot \Delta w(p_0, 1)$, we need to show

$$\frac{2 \cdot (1 - u)}{2 - u} > \frac{1}{e}. \tag{A156}$$

In this case, $u = 0.722$ can be calculated numerically by plugging $p_0 = \frac{2}{\lambda}$ and $\beta = 1$ into (A150) and solve the non-linear equation:

$$1 - u = \exp(-2 + u). \tag{A157}$$

Both sides of (A156) can then be calculated, with the left-hand-side equaling 0.435 and the right-hand-side equaling 0.368. □

B Distributions Fulfilling Assumption 1 (Reciprocal of Survival Function is Convex)

This appendix catalogs common distributions that meet Assumption 1 (reciprocal of the survival function is convex). This is a weak form of log-concavity of a distribution and is met by most commonly-used distributions.

B.1 Definitions

Probability density function:

$$f(x) \tag{B.1}$$

Cumulative density function:

$$F(x) \tag{B.2}$$

Survival function:

$$G(x) = 1 - F(x) \tag{B.3}$$

B.2 Conditions

Assumption 1 in the main text (reciprocal of the survival function is convex) is the weakest of three types of log-concavity for distributions. We examine all three types since if one of the stricter types of log-concavity is met, then the weaker forms are also met and these are sometimes more straightforward to show. The three types of log-concavity in decreasing order of strictness are:

Condition 1: Concavity of survival function: $G(x)$ is concave ($F(x)$ is convex) in x, which requires:

$$f'(x) > 0. \tag{B.4}$$

Condition 2: Log-concavity of survival function: $log(G(x))$ is concave, which implies $F(x)$ has a hazard rate:

$$h(x) = \frac{f(x)}{1 - F(x)}, \tag{B.5}$$

that is increasing in x:

$$h'(x) > 0. \tag{B.6}$$

Condition 3 (corresponding to Assumption 1 in the main text): Reciprocal of the survival function is convex. $w(x) = G(x)^{-1} = \frac{1}{1-F(x)}$ is convex:

$$w''(x) = \frac{f'(x)}{(1 - F(x))^2} + \frac{f(x)^2}{(1 - F(x))^3} > 0, \tag{B.7}$$

or:

$$f'(x)(1 - F(x)) + f(x)^2 > 0. \tag{B.8}$$

B.3 Distributions

We verify Assumption 1 is met for the following distributions:

1. Uniform
2. Exponential
3. Normal
4. Beta
5. Pareto
6. Generalized Extreme value (Weibull distribution)
7. Chi-squared

B.4 Checking Conditions

B.4.1 Uniform Distribution

The pdf is:

$$f(x; u, l) = \begin{cases} \frac{1}{u-l} & \text{for } x \in [l, u] \\ 0 & \text{otherwise.} \end{cases} \tag{B.9}$$

The cdf is:

$$F(x; u, l) = \begin{cases} 0 & \text{for } x < l \\ \frac{x-l}{u-l} & \text{for } x \in [l, u] \\ 1 & \text{for } x > u. \end{cases} \tag{B.10}$$

The hazard function is defined on $[l, u]$ as:

$$h(x; u, l) = \frac{1}{u-x}. \tag{B.11}$$

Condition 1 is not met:

$$f'(x; u, l) = 0. \tag{B.12}$$

But Condition 2 is met:

$$h'(x; u, l) = \frac{1}{(u-x)^2} > 0. \tag{B.13}$$

B.4.2 Exponential Distribution

The pdf is:

$$f(x; \lambda) = \lambda exp(-\lambda x), \tag{B.14}$$

where $\lambda > 0$. The cdf is:

$$F(x; \lambda) = 1 - exp(-\lambda x). \tag{B.15}$$

The hazard function is:

$$h(x; \lambda) = \lambda. \tag{B.16}$$

The reciprocal of the survival function is:

$$w(x; \lambda)^{-1} = exp(\lambda x). \tag{B.17}$$

Condition 1 is not met:

$$f'(x; \lambda) = -\lambda^2 exp(-\lambda x) < 0. \tag{B.18}$$

Condition 2 is also not met:

$$h'(x; \lambda) = 0. \tag{B.19}$$

But Condition 3 is met:

$$w''(x;\lambda) = \lambda^2 exp(\lambda x) > 0. \tag{B.20}$$

B.4.3 Normal Distribution

The pdf is:

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}. \tag{B.21}$$

An alternative way to verify Condition 1 is to check if the log of the pdf function is concave (Bagnoli and Bergstrom (2005)). This confirms Condition 1:

$$log\,[f(x;\mu,\sigma)]'' = -\frac{1}{\sigma} < 0. \tag{B.22}$$

B.4.4 Beta Distribution

The pdf is:

$$f(x;\alpha,\beta) = \frac{1}{\mathcal{B}(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{B.23}$$

where $0 < x < 1$, $\mathcal{B} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, and Γ is the Gamma function.

Condition 1 is met if $\alpha > 1$ and $\beta > 1$ since:

$$f'(x;\alpha,\beta) = \frac{1}{\mathcal{B}(\alpha,\beta)}(\alpha-1)x^{\alpha-2}(1-x)^{\beta-1} + (\beta-1)x^{\alpha-1}(1-x)^{\beta-2} > 0. \tag{B.24}$$

B.4.5 Pareto Distribution

The pdf is:

$$f_X(x;\alpha,x_m) = \begin{cases} \frac{\alpha {x_m}^{\alpha}}{x^{\alpha+1}} & \text{if } x \geq x_m \\ 0 & \text{if } x < x_m, \end{cases} \tag{B.25}$$

where $\alpha > 0$ and $x_m > 0$. The cdf is:

$$F(x;\alpha,x_m) = \begin{cases} 1 - (\frac{x_m}{x})^{\alpha} & \text{if } x \geq x_m \\ 0 & \text{if } x < x_m. \end{cases} \tag{B.26}$$

The survival function is:

$$G(x;\alpha,x_m) = (\frac{x_m}{x})^{\alpha}. \tag{B.27}$$

The hazard function is:

$$h(x;\alpha,x_m) = \frac{\alpha}{x}\,, \tag{B.28}$$

where $x > x_m$. Condition 1 is not met since:

$$f'(x;\alpha,x_m) = -\alpha(\alpha+1)x_m^{\alpha}x^{-\alpha-2} < 0. \tag{B.29}$$

Condition 2 is not met since:

$$h'(x;\alpha,x_m) = -\alpha x^{-2} < 0. \tag{B.30}$$

However, Condition 3 is met since:

$$w''(x;\alpha,x_m) = \alpha(\alpha+1)x_m x^{-\alpha-2} > 0. \tag{B.31}$$

B.4.6 Generalized extreme value distribution (Weibull)

The pdf is:

$$f(x;c) = cx^{c-1}e^{-x^c}, \tag{B.32}$$

where $x > 0$ and $c \geq 1$. The cdf is:

$$F(x;c) = 1 - e^{-x^c}. \tag{B.33}$$

The survival function is:

$$G(x;c) = e^{-x^c}. \tag{B.34}$$

and the hazard rate is:

$$h(x;c) = cx^{c-1}. \tag{B.35}$$

Condition 1 is met only when $x < c - 1$:

$$f'(x;c) = c(c-1)x^{c-2}e^{-x^c} - cx^{2c-2}e^{-x^c} > 0. \tag{B.36}$$

However, Condition 2 is met for all values of x:

$$h'(x;c) = c(c-1)x^{c-2} > 0. \tag{B.37}$$

B.4.7 Chi-squared Distribution

The pdf is:

$$f(x;k) = \frac{1}{2^{\frac{k}{2}}}\Gamma(\frac{k}{2})x^{\frac{k}{2}-1}e^{-\frac{x}{2}}, \tag{B.38}$$

where Γ denotes the Gamma function and $k \geq 2$.

We can verify the log of the pdf function is concave to confirm Condition 1 (Bagnoli and Bergstrom (2005)):

$$log\left[f(x;k)\right]'' = -\left(\frac{k}{2} - 1\right)\frac{1}{x^2} < 0. \tag{B.39}$$